MAH/MUL/ 03051/2012 ISSN :2319 9318

आंतरविद्याशाखीय बहुभाषिक शोध पत्रिका

विद्यावार्ता

One Day National Interdisciplinary Conference 28 April 2023

Organized by Rayat Shikshan Sanstha'sProf. Dr. N.D. Patil Mahavidyalaya, Malkapur (Perid)

Rayat Shikshan Sanstha's

Prof. Dr. N. D. Patil Mahavidyalaya, Malkapur (Perid)

One Day National Interdisciplinary Conference

On

Literary Trends and Issues in 21st Century

Friday, 28 April 2023

Organized by

Departments of Marathi, Hindi, English, and IQAC

Chief Editor: - Dr. Gholap Bapu Ganpat

"Printed by: **Harshwardhan Publication Pvt.Ltd.** Published by **Ghodke Archana Rajendra** & Printed & published at Harshwardhan Publication Pvt.Ltd.,At.Post. Limbaganesh Dist,Beed -431122 (Maharashtra) and Editor **Dr. Gholap Bapu Ganpat.**

Reg.No.U74120 MH2013 PTC 251205

Harshwardhan Publication Pvt.Ltd.

At.Post.Limbaganesh,Tq.Dist.Beed
Pin-431126 (Maharashtra) Cell:07588057695,09850203295
harshwardhanpubli@gmail.com, vidyawarta@gmail.com

All Types Educational & Reference Book Publisher & Distributors / www.vidyawarta.com

२) मध्ययुगीन कालखंडातील साहित्याने मूल्यांचा पाया घातला परंतु स्वातंर्त्योत्तर काळातील साहित्यातून मूल्यांचा विकास अधिक होणे आवश्यक होते तसे चित्र दिसत नाही.

३) मानवी जीवन विकसित व यशस्वी करण्याची क्षमता मूल्यांमध्ये आहे. शाश्वत मूल्यांची पखरण नवोदित साहित्यातून होणे गरजेचे आहे.

४) साहित्यातून मानवी जीवनमूल्य जपली जात नसतील तर ते साहित्य समृद्ध आहे असे म्हणणे निरर्थक: ठरते.

५) मानवी जीवनात मूल्यांना फार महत्त्व आहे ते साहित्यिकांनी आपल्या साहित्यातून जोपासणे तितकेच महत्त्वाचे आहे.

संदर्भग्रंथ

१.कोत्तापल्ले, डॉ. नागनाथ, 'आधुनिक मराठी कविता एक दृष्टिक्षेप' प्रतिभास प्रकाशन, परभणी, २००७.

२. राऊत, डॉ. अनंत, 'मूल्यनिष्ठ समाजघडणीत साहित्याची भूमिका', कैलास पब्लिकेशन्स, औरंगाबाद, प्रथमावृत्ती २६ जानेवारी २००८.

३. गोसावी, र. रा. 'पाच भक्तीसंप्रदाय', प्रतिमा प्रकाशन, पुणे तृतीयावृत्ती, १५ ऑगस्ट २००८

४. खैरनार, डॉ. दिलीप, 'भारतीय समाजातील नैतिक मूल्ये', डायमंड पब्लिकेशन्स, पुणे, प्रथमावृत्ती २००६.

५.गायकवाड,लक्ष्मण . वडार वेदना,प्रथमावृत्ती २०००

⌘⌘⌘

30

'निंभार': आत्मनिष्ठ जाणिवेचा आविष्कार

प्रा. संजय शंकर सुतार
मराठी,विभागप्रमुख,
दत्ताजीराव कदम आर्ट्स, सायन्स अण्ड कॉमर्स कॉलेज, इचलकरंजी

डॉ. ज्ञानेश्वर रामचंद्र कांबळे
सहायक प्राध्यापक,
दत्ताजीराव कदम आर्ट्स,सायन्स अण्ड कॉमर्स कॉलेज, इचलकरंजी

****

प्रत्येक व्यक्तीच्या अंतर्मनात एखादा लेखक दडलेला असतो. एखाद्या पाषाणातून पाझर फुटावा व त्या उमाळ्यातून ओलावा निर्माण होऊन त्यातून अंकुर फुटावेत अशा अंत:प्रेरणेतून लेखक आपलं जगण साहित्याच्या परिघात मांडत असतो. अशा अनुभवसंपन्न आशय विचारामधून साहित्यकृती जन्म घेते. अशी साहित्यकृती त्या वाङ्मयप्रवाहातील मैलाचा दगड म्हणून ठसठशीतपणे उभी राहते. तिचे अस्तित्व त्या साहित्य प्रकारातील नवे संदर्भमूल्य म्हणून अधोरेखित होते.या अनुषंगाने साहित्य जगतात 'दीर्घ कथनात्मक गद्याचा आविष्कार म्हणून कादंबरीकडे पाहिले गेले आहे'. कादंबरी हा मुद्रणोत्तर जगतातील आधुनिक साहित्यप्रकार म्हणून ओळखला गेला आहे. या साहित्य प्रकारातून व्यक्ती आणि समाज यांच्यामधील नातेसंबंधांची वीण प्रामुख्याने गुंतलेली असते. त्या नात्यासंबंधाच्या भोवतालच्या परिघात घडलेल्या स्थित्यंतराचे वैश्विक रूप कादंबरीतून प्रकटत जाते. पाश्चात्य जगात आधुनिकतेमधून निर्माण झालेल्या नवमध्यमवर्गातून आकारास आलेल्या या वाङ्मय प्रकारात जागतिकीकरणानंतर मानवी जीवनावर झालेल्या अनुकूल—प्रतिकूल बदलांकडे समग्र दृष्टीने लक्ष

वेधले आहे. तद्नंतर झालेल्या वैश्विक बदलांकडे विविध साहित्य प्रवाहांप्रमाणे कादंबरीच्या माध्यमातूनही मानवी भावभावना, माणूस व समाजामधील नातेसंबंधांची गुंतागुंत, त्यासह भोवतालच्या वास्तवाच्या विविध परि आणि सीमा अधोरेखित करीत भाषिक प्रयोगाच्या आणि अर्थाच्या शक्यता कादंबरीमधून पडताळले गेलेले दिसून येतात.

काळाच्या प्रत्येक टप्प्यांवर बदलत्या स्थित्यंतरानुसार 'वर्तमानाचे नवे रचित साकारणारी कादंबरी नवी भाषा, नवे रचनाबंध ,नवी मूल्ये रुजवण्याचे आणि वाचकांची अभिरुची घडवण्याचे काम करत आहे.' विशेषत: एकविसाव्या शतकातील बहुतांश कादंबऱ्यांमधून सिमित अवकाशाच्या पलीकडे नवी परिमाणे शोधत आधुनिक जगातील बदलत्या मूल्यदृष्टीचा वेध घेत एक नवी जीवनदृष्टी साहित्यातून प्रकट झालेली दिसते. त्यामध्ये प्रामुख्याने 'बारोमास', 'भूमी', 'गाथा सप्तपती' , 'संभुती','वारुळ', 'चाळेगत', 'इळनमाळ', 'शाळा', 'आगळ','अवकाळी पावसाच्या दरम्यानची गोष्ट', 'जोहार', 'नवं कटवन','पोटमारा', 'सायड', 'अवघाचि संसार', 'आटपाट देशातल्या गोष्टी', 'स्वत:ला फालतू समजण्याची गोष्ट', 'उद्या', 'रौंदाळा', 'पडझड', धुळपावलं', 'बगळा', 'निशाणी डावा अंगठा','लॅंडमाफिया','रिबोट', 'रहबर', 'जुगाड','उजव्या सोंडेच्या बाहुल्या', 'झडझिंबड','गाज','ईश्वर डॉट कॉम', इत्यादी प्राथमिक कादंबऱ्यांमधून भोवतालच्या काळाचे संदर्भ शोधत मानवी जीवनाच्या अवस्थांतरांचे प्रतिबिंब शोधण्याचा जाणीवपूर्वक प्रयत्न केल्याचे दिसून येते. या स्थित्यंतराच्या कालौघात सामान्य माणूस त्याच्या उध्वस्तपणाचे, रितेपणाचे, गारठलेल्या संवेदनाचे, शेतीच्या अधोगतीतून चक्रव्यूहात सापडलेल्या कृषिसंस्कृतीचे विदारक सत्य मांडताना माणसाचे एकाकीपण संस्कृतीचा पोकळ बडिवार,सामाजिक संबंधातील बदलते संदर्भ, मानवी जीवनातील अतर्क्य ,असंगततेला स्पर्श करणारी परिस्थिती यांचे वेगळेपण अधोरेखित करणाऱ्या वरील कादंबऱ्यांच्या वर्तुळविश्वात 'निंभार' ही आशय आकाराचे अभिनत्व मांडणारी एक प्रयोगशील कादंबरी म्हणून तीचे महत्त्व अधोरेखित होते. वास्तवाच्या अंतरंगात जाउन जे जाणवले त्याचा आत्मसाक्षात्कार करताना दिसून येते. 'निंभार' म्हणजे मध्यान्हीचे तळपतं ऊन. प्रत्येक व्यक्तीच्या आयुष्यात ऊन सावल्यांचा खेळ चालत असतो. धगधगत्या उन्हात होणारी तगमग ही अस्वस्थता निर्माण करते. ही अस्वस्थता लेखक संतोष तेंडुलकर यांच्या लेखन सामग्रीचे आत्मबळ म्हणून पुढे येते आणि त्यातून 'निंभार' सारख्या प्रयोगशील कादंबरीची रचनानिर्मितीची प्रक्रिया घडते.

जीवनानुभवाचे 'समृद्ध स्तर मांडणारी प्रस्तुत कादंबरी समाजजीवनाच्या अनेकस्तरीय वाटा धुंडाळताना दिसते.'' या कादंबरीचा निवेदक हा ताऊन सुलाखून निघालेल्या परिस्थितीवर मात करीत शिक्षणाची उमेद जागविण्याचे आत्मभान जपतो. कोकणातल्या कोंडये गावाच्या केंद्रावर्ती घटनांचा पसारा मांडताना कोकणातील नैसर्गिक स्थित्यंतरे कौटुंबिक, सामाजिक अस्थिरता यामधून शिक्षणासाठीची झालेली धावाधाव या सर्वांच्या रेट्यतून निवेदक आपली इच्छाशक्तीची उमेद जपून ठेवतो. माध्यमिक शिक्षण घेत असताना घडत गेलेल्या घटनाक्रमामधून मॅकेनिकल डिप्लोमा झालेला निवेदक सौदी अरेबिया येथील तैवान मधील मिळालेल्या संधीमधून स्वत:ला सिद्ध करण्याचा प्रयत्न करतो. येणाऱ्या प्रत्येक संकटांशी सामना करीत स्वत:ला घडवत जातो. तेथे वास्तव्यास असताना आंतरराष्ट्रीय देशातील सांस्कृतिक संदर्भ तेथील व्यक्तिस्वातंर्त्य, संस्कृति—सभ्यता, परंपरा, स्त्री—पुरुष विचारधारा या सर्वामध्ये भारत देश कसा उजवा आहे . याची प्रचिती या निमिताने निवेदकाने कादंबरीच्या माध्यमातून स्वानुभवातून विस्तृतपणे रेखाटली आहे. तैवान मधील भाषा, लोकविचार, भाव भावना, आचार—विचार यातील प्रस्तुत कादंबरीच्या रचना सौंदर्या बरोबर भाषा— सौंदर्यात वेगळेपणाची अनुभूती देते. व्यावसायिक जीवनदृष्टिकोनासह लेखकाने गावखेड्यच्या वैशिष्ट्यंचा माणसांचा, तेथील ग्रामीण जीवनाचे दिलेले संदर्भ कादंबरीत विशेषत्वाने नजरेत भरते.

कोकणातल्या गतस्मृतींना उजाळा देणारा निवेदक कादंबरीच्या माध्यमातून त्यामध्ये विरघळून जातो. कोकणातल्या भोवतालात आपले बालपण शोधताना तिथल्या नैसर्गिक वातावरणातील संस्कृतिविशेषामधील झालेले बदल अत्यंत मार्मिक

पद्धतीने टिपले आहेत. कोकण म्हणजे निसर्ग, समुद्र आणि फणसासारखी माणसे त्यांच्या व्यथा वेदनांचे मनोरे लेखकाने कादंबरीत उभारलेले आहेत. कोकणी माणसांमधला चिकित्सकपणा त्याच्यामधील काटकपणा या स्वाभाविक वैशिष्ट्यंमधून अनेक पात्रांमधून वेगळेपण दृष्टीक्षेपात येते. विविध प्रसंगामधून अनेक वृत्ती—प्रवृत्तीच्या माणसांमुळे 'निवेदकाचे जीवन बदलण्यास ही माणसं कारणीभूत झालेली दिसतात. निवेदकाच्या वडिलांनी आयुष्यभर भली—बुरी माणसं जपली व वाढवली पण नंतरच्या काळात त्यांनी सोडलेली साथ ही विचार करण्यास प्रवृत्त करते.

वरील मानवी वैशिष्ट्यंखेरीज कोकणातील पावसाळ्यापूर्वी व नंतरचे जीवन, निसर्गाच्या प्रतिकूलतेनुसार कोकणी माणसाची वाटचाल खाचे विशेष कादबरीतून पुढे येते. यामध्ये मसाल्याचे काम करणारी सखू सोनारीन, उबदार गोधड्य शिवणारी जनाबाई. पाडाला आलेले रायवळ आंबे उतरुन आडी लावणारा आबा शिवगण, भातलागणीच्या वेळी चिखलावर काटेरी गुफा (दाता) फिरवणारा वासु व त्यावर उभा राहणारा नाना परब, अनैतिकतेच्या चक्रव्यूहात सापडलेला गोपाळ तसेच निवेदकाच्या घरी गोठबतील जनावरांची काळजी घेणारा पांडू या शिवाय लेखकाचं जीवन बदलण्यास कारणीभूत ठरलेली आई, वडील, भाऊ व बना मावशी, सर्व भावंड ही पात्रे निवेदकाच्या जगण्यावर खोलवर परिणाम करताना दिसून येतात.

प्रस्तुत कादंबरीतील ही जीवाभावाची माणसं हीच निवेदकाच्या जगण्याला अर्थ प्राप्त करून देणारे घटक आहेत. इथल्या सामान्य माणसांमध्ये देवपण शोध णारा नायक कोकणातल्या रिती, परंपरा संस्कृतिसंघर्ष, उत्सवातील वेगळेपण या कादंबरीच्या रुपाने मांडत जातो. प्रत्येक सण उत्सवातील कोकणी माणसांनी जपलेलं वैभव भातशेतीच्या दरवळाप्रमाणे खऱ्याअर्थाने कादंबरीतून पाझरताना दिसते. 'कादंबरीकार अशा घटनांची निवड आणि मांडणी करीत असतो. ज्यांच्या द्वारा भोवतीच्या विश्वातील माणसे, त्यांचे परस्पर संबंध व सामाजिक संस्था व त्यांची तत्वे यावर प्रकाश पडावा तसेच त्या विशिष्ट काळातील सामाजिक व नैतिक मूल्ये आणि तो वर्णन करीत असलेल्या कल्पित विश्वातील सामाजिक वा नैतिक मूल्ये यांच्याशी वाचक कुठेतरी नाते जोडू शकेल' या भूमिकेतून लेखक संतोष तेंडुलकरांनी निवेदकाच्या जीवनव्यवहाराशी आपली नाळ कायम ठेवत वाचकाला प्रेरित केले आहे. कोकणातून स्थलांतर होणारे चाकरमानी त्यांचे प्रश्न याशिवाय मुंबईत राहणाऱ्या सामान्य माणूस हा केवळ ग्राहक' याच केंद्रस्थानी राहिल्याने त्यामुळे निर्माण झालेल्या परिस्थितीचे आत्मचिंतन कादंबरीच्या वर्तुळ परीघातून घडताना जाणवते. साहजिकच आधुनिकतेच्या जगतात नवमूल्यांची होणारी हेळसांड या कादंबरीच्या रूपाने समोर येते. प्रस्तुत कादंबरीच्या निवेदकाच्या वाट्यला आयुष्यभर संघर्ष आला. त्यामधून स्वतःला शोधताना आपल्या आशा—आकांशा याचं निर्माल्य करावं लागलं. तरीही उद्याच्या नव्या दिवसाचा शोध घेणारा निवेदक प्रेरणास्त्रोत म्हणूनच पुढे येताना जाणवतो. जीवनाच्या या कोलाहलात स्वत्व जपताना आई—वडील हीच उर्जा निवेदकाच्या प्रत्येक प्रतिकूल परिस्थितांशी संघर्ष करण्यासाठी बळ देते. कुटुंब जपणारी आई ही कर्त्या पुरुषाची जागा घेताना दिसते. तर आजारपणात वडिलांची काळजी घेणारी कुटुंबवत्सल आई मध्ये लेखक श्यामच्या आईचे रूप शोधताना दिसतो. आयुष्यभर कष्टणारी आई बरोबरच ज्यांनी गावाला 'दिशा' दिली व गाव जपला त्या वडिलांनी समाजाला आधुनिक विचार देण्याचा प्रयत्न केला. सतत नव्या उद्योग व्यवसायातून वेगळेपण शोधण्याचा ध्यास घेतला. पण लोकांनी त्यांच्या पडत्या काळात पाठ फिरविरली ही अस्वीकाहार्य गोष्ट लेखकाच्या अंतर्मनाला छेदून टाकते. समाज व्यवस्थेतील वेगाने होणारे बदल लेखकाने सूक्ष्म तपशिलांसह मांडत हरवत चाललेल्या मानवी मूल्यांकडे लक्ष वेधले आहे. आयुष्यभर कष्टमय जीवन जगलेल्या निवेदकाच्या वडिलांच्या वाट्यला नियतीने लिव्हर कॅन्सरचे व्याधीग्रस्त जगणे बहाल करून निवेदकाला पुन्हा चक्रव्यूहात टाकले. आजारपणात त्यांची शुश्रूषा घेणाऱ्या व जीवापाड प्रेम करणाऱ्या नायकाच्या वाट्यला शेवटी अत्यंदर्शन घ्यावे लागले अन् त्याच वेळी काही अंतराने गेलेल्या आईच्या निधनाने दुहेरी संकटात नायकाच्या भोवती आभाळ कोसळते.

सारं उसवतं आयुष्य आम्ही टाके हे घालतो
इथं वैशाख वणवा आम्ही पाऊस शोधतो

अशा संघर्षमयी जीवनाच्या विविधांगी कोनांमधून लेखक आपल्या अस्तित्वाचा खुणा शोधताना दिसून येतात.

प्रस्तुत कादंबरीत मराठी, हिंदी मिश्रित व कोकणी भाषेची रूपे आविष्कृत झालेली दिसतात याशिवाय कोकणात नाव घेऊन बोलावण्याची पद्धती त्यामधील गोडवा, प्रादेशिक म्हणीमधून कादंबरीच्या भाषा सौंदर्यात भर पडलेली दिसून येते. संतोष तेंडुलकर व लिखित या आत्मनिवेदनपर कादंबरीत व्यवधनांमधून समाज, गावखेडे, शोषणव्यवस्था, शहरीकरणाच्या समस्या, मुंबईतील स्थित्यंतरे, जागतिक संदर्भातील भारत या सर्वांचे परि आणि सीमा यांना अधोरेखित करण्याचा यशस्वी प्रयत्न केला आहे. ज्या पद्धतीने भालचंद्र नेमाडे यांच्या 'बिढार,' 'जरीला', 'झूल' तसेच विलास सारंग यांच्या 'एन्कीच्या राज्यात', भाऊ पाध्ये यांची 'बॅरिस्टर अनिरुद्ध धोपेश्वर',किरण नगरकर यांची 'सात सक्कं त्रेचाळीस', दीनानाथ मनोहर यांच्या 'मन्वंतर' प्रमाणे 'निंभार' मधून माणसांच्या दैनंदिन जीवन व्यवहारांमधील अनाकलनीयता, असंगतता, एकाकीपणाची आत्मजाणीव व त्यातील अस्वस्थता यामधील आत्मनिष्ठ जाणिवेतून स्व—शोधार्थ निघालेल्या लेखकाने भोवतालचा जगाचे दर्शन घडविले आहे. अशा काल व अवकाशाचा विस्तृत पटात अस्तित्वाचा शोध घेणारी ही कादंबरी संकल्पनात्मक बदलांकडे लक्ष वेधून घेते. तसेच मानवतेच्या संचिताचे भाष्य करणारी कादंबरी म्हणून 'निंभार' तळपत्या मध्यानहीच्या उन्हाप्रमाणे विचारांचे 'काहूर निर्माण करण्यात यशस्वी होईल अशी आशा वाटते.

संदर्भ ग्रंथसूची :

१) तेंडुलकर, संतोष 'निंभार', तेजश्री प्रकाशन, कबनूर, २०२३.

२) थोरात, हरिश्चंद्र 'कादंबरी विषयी' पद्मगंधा प्रकाशन, पुणे, २००६.

३) इनामदार—साने, रेखा' अस्तित्ववाद आणि मराठी कादंबरी', राजहंस प्रकाशन,पुणे, २००४.

४) जाधव,रा.ग (संपा.)' मराठी वाङ्मयाचा इतिहास' खंड— सातवा, भाग तिसरा, महाराष्ट्र साहित्य परिषद, पुणे, २०१०.

५) सानप, किशोर 'मराठी कादंबरी नव्या दिशा' निर्मल प्रकाशन, नांदेड, २००५.

६) घोंगडे,बाळासाहेब' अक्षर पेरणी — कादंबरी निर्मिती प्रक्रिया' विशेषांक' ऑक्टोबर —नोव्हेंबर—डिसेंबर, २०१६.

७) थोरात,हरिश्चंद्र 'कादंबरीः एक साहित्यप्रकार' , शब्द पब्लिकेशन, मुंबई.२०१०.

८) काळे, मंगेश नारायणराव '१९६० नंतरचे बदलते संदर्भ आणि मराठी कादंबरी', 'खेळ' नव्वदोत्तर कादंबरी विशेषांक, जून २०१४.

९) गणोरकर, प्रभा आणि इतर (संपा.) वाङ्मयीन संज्ञा संकल्पना कोश,ग.रा. भटकळ फाउण्डेशन, मुंबई, २००१.

१०) राज्याध्यक्ष, विजया 'मराठी वाङ्मयकोश', समीक्षा — संज्ञा, खंड —चौथा, महाराष्ट्र राज्य साहित्य आणि संस्कृती मंडळ,मुंबई. २०११.

⌘⌘⌘

INTERNATIONAL RESEARCH JOURNAL OF HUMANITIES AND INTERDISCIPLINARY STUDIES
(Peer-reviewed, Refereed, Indexed & Open Access Journal)
DOI : 03.2021-11278686 ISSN : 2582-8568 IMPACT FACTOR : 6.865 (SJIF 2023)

Differential Analysis of Search Engine Optimization

Dr. Sulbha Gath
Dattajirao Kadam Arts, Science and Commerce College,
Ichalkaranji, Dist. Kolhapur (Maharashtra, India)
E-mail: gathsjj@gmail.com

DOI No. 03.2021-11278686 **DOI Link ::** https://doi-ds.org/doilink/03.2023-16744626/IRJHISIC2302040

Abstract:

In today's world, the search engine is essential to E-marketing. We receive the search results from the search engine. The outcomes are displayed as a list. Optimizing the search result is a search engine's main objective. Based on relevance, the search engine optimizes the search results. Based on several factors, search engines give a given web page a rank. Elements including link popularity, keywords, etc. To obtain the most pertinent search results, one can take into account several search engine strategies such as on-page and off-page search engine optimization (SEO). In this essay, we attempt to examine the well-known search engine optimization approaches. We also examine several forms of search engine optimization.

Keywords: *SEO, White-hat SEO, Black-hat SEO, Gray-hat SEO, On-page SEO, Off-page SEO.*

Introduction:

Differential analysis of search engine optimization (SEO) is a method of comparing two or more websites or web pages to identify differences in their (Nurmi)SEO strategies and tactics. This can be done by analysing various elements of the websites such as their content, structure, meta tags, and inbound links. By identifying these differences, it is possible to gain insights into what strategies and tactics are working well for the websites being analysed, and to identify areas where improvements can be made to your own website.

There are various tools and techniques that can be used to perform a differential analysis of SEO. Some of these include:

White-hat SEO:

White Hat SEO refers to the use of ethical and legitimate techniques to improve a website's search engine ranking. These techniques align with the search engines' guidelines and aim to provide a positive user experience. White Hat SEO techniques include:

Black Hat SEO:

Black Hat SEO refers to the use of unethical or manipulative techniques to improve a website's search engine ranking. (Couzin, 2008)These techniques are in violation of search engine guidelines and can result in penalties or even being banned from the search engine's index.

Gray-hat SEO:

Gray Hat SEO refers to techniques that fall in between White Hat and Black Hat SEO. These techniques are not strictly in violation of search engine guidelines, but they are not necessarily in compliance with them either.

Type Of SEO Techniques:

There are four main types of SEO techniques: on-page SEO and off-page SEO, Technical SEO:

1. **On-page SEO:** refers to the techniques used to optimise a website's content and structure to improve its search engine ranking. (Kent, 2009)This includes optimising the website's title tags, meta descriptions, header tags, and content for relevant keywords. It also includes techniques such as improving the website's loading speed and mobile responsiveness, and ensuring that the website is accessible to search engines through the use of proper site architecture and internal linking.
2. **Off-page SEO:** refers to the techniques used to improve a website's search engine ranking through external sources. This includes building backlinks from other websites to your own, as well as building a strong online presence through social media and other forms of online marketing. Off-page SEO also includes building the brand reputation and trust.
3. **Technical SEO:** refers to the optimization of a website's underlying code, structure, and server settings in order to improve its visibility and accessibility to search engines. (Munsell, 2010)This includes making sure that the website is properly indexed, that it loads quickly, and that it is mobile-friendly. Technical SEO also involves ensuring that the website's URLs are clear and descriptive, that there are no broken links or 404 errors, and that the website is free from any penalties or errors that could impact its visibility in search engine results.

SEO audit:

is a process of analyzing and evaluating a website's technical and on-page SEO performance in order to identify any issues that may be impacting its search engine rankings. The goal of an SEO audit is to identify areas that need improvement and to provide recommendations for how to optimize the website for search engines (White, 2011).

SERP can be affected by different ranking factors, and can be influenced by different SEO techniques. Some factors include, keyword relevance, back links, meta tags, and many others. SERP can vary depending on the search engine being used, as well as the user's location and search history. SERP can also include various types of content such as featured snippets, map packs, local packs, images, videos, and many others, which can be used to help users find the most relevant and useful information for their search query.

Google Tools For SEO:

#Google Search Console: Google Search Console is a free tool offered by Google that allows webmasters to monitor and maintain their site's presence in Google search results. It provides information and insights about how your website is performing in search results.

#Google Analytics: Google Analytics is a free web analytics service offered by Google that tracks and reports website traffic. It helps website owners understand how visitors interact with their website.

#Google Ads: Google Ads (formerly known as Google AdWords) is a pay-per-click (PPC) advertising platform that allows businesses to display ads to users on Google search results and other websites.

#Google Trends: Google Trends is a free tool offered by Google that allows users to explore data and trends about specific search terms and topics. It provides information about the popularity of a search term over time, as well as related queries and topics, and the geographic locations where the search term is most popular (James).

Google's algorithms:

Google's algorithms are complex mathematical formulas that are used to determine the relevance and importance of web pages for specific search queries. These algorithms help Google to deliver the most relevant and high-quality search results to users. Here are some of the key algorithms used by Google:

1. **PageRank:** PageRank was Google's original algorithm that was used to rank web pages based on their importance and relevance. It uses a complex formula that takes into account the number and quality of links pointing to a page, among other factors.
2. **Hummingbird:** Hummingbird is a more recent algorithm that was designed to better understand the meaning behind a user's search query and to deliver more relevant results. It focuses on the context of the search query, rather than just individual keywords.
3. **Penguin:** The Penguin algorithm was designed to penalize websites that engage in spammy link building tactics, such as buying links or participating in link schemes (Carson, 2013).
4. **Panda:** The Panda algorithm was designed to penalize websites that have low-quality or thin

content. It's intended to reward high-quality websites and to improve the overall quality of search results.

5. **BERT:** BERT is a language processing algorithm that helps Google to better understand the context and meaning of words and phrases in a user's search query. It's designed to deliver more accurate and relevant results for natural language queries.

These are just a few of the many algorithms used by Google to determine the relevance and importance of web pages for specific search queries. Google's algorithms are constantly evolving and being updated, so it's important for websites to stay up-to-date with best practices for search engine optimization (SEO) to maintain their visibility in search results (Johns, 2014).

Mobile and desktop SEO:

Mobile and desktop SEO are similar in many ways, but there are also some key differences to consider. Here are the main differences between mobile and desktop SEO:

- Device compatibility: Mobile SEO needs to take into account the various types of mobile devices, screen sizes, and browsers, whereas desktop SEO just needs to worry about desktop computers.
- User experience: Mobile devices have smaller screens and touch-based navigation, which can affect the user experience. Mobile SEO needs to optimize for this experience, such as using a responsive design and larger font sizes.
- Load speed: Mobile devices often have slower internet connections, so mobile SEO needs to prioritize fast load times to keep users engaged.
- Local search: Mobile devices are often used on-the-go, so mobile SEO needs to prioritize local search results and maps.
- Voice search: Voice search is becoming increasingly popular on mobile devices, so mobile SEO needs to optimize for natural language queries.
- App indexing: Mobile devices often use apps, so mobile SEO needs to consider app indexing to ensure that app content can be found in search results.

While both mobile and desktop SEO need to focus on providing relevant and high-quality content, optimizing for keywords, and building links, there are important differences to consider for each device type. A comprehensive SEO strategy needs to take both mobile and desktop into account to reach the maximum number of users (Cutts, 2015).

Search behaviour:

Search behaviour refers to the actions and patterns of users when they search for information online using search engines such as Google, Bing, Yahoo, etc. Understanding search behaviour can help companies to improve their search engine optimization (SEO) and search engine marketing

(SEM) strategies, as well as their overall online presence.

By understanding search behaviour, companies can improve their search rankings and reach more users with relevant and targeted content. They can also use this information to optimize their websites for specific devices and improve the user experience for their target audience.

Ranking Algorithm:

A ranking algorithm is a mathematical formula used by search engines to determine the relevance and importance of web pages for a specific search query. The ranking algorithm takes into account various factors such as keywords, content quality, user engagement, and backlinks, to determine the relevance of a web page to a search query. The ultimate goal of a ranking algorithm is to provide the most relevant and high-quality search results to users (Enge, 2016).

Here are some of the key factors that a ranking algorithm might consider:

1. Keywords: The presence and relevance of keywords in the title tag, headings, and body content of a web page can impact its ranking.
2. Content quality: Search engines look for high-quality, unique, and relevant content that provides value to users. Pages with low-quality, duplicate, or thin content may be penalised.
3. User engagement: User engagement metrics such as click-through rate (CTR), time on site, and bounce rate can impact a web page's ranking. Pages with high engagement signals are often seen as more relevant and authoritative.
4. Backlinks: The number and quality of backlinks pointing to a web page can impact its ranking. Pages with a high number of high-quality backlinks are often seen as more authoritative and trustworthy.
5. Mobile optimization: With the increasing popularity of mobile devices, search engines now place a high importance on mobile optimization. Pages that are not mobile-friendly may be penalised in search results.
6. Local optimization: For local search queries, search engines consider factors such as the location of the business, distance from the user, and presence in local directories (Fishkin, 2017).

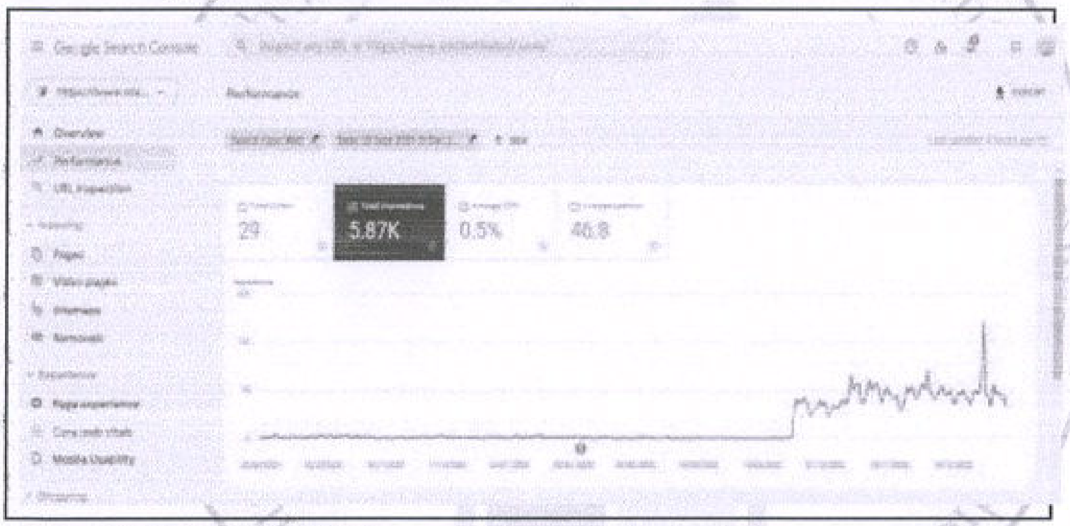
These are just a few of the many factors that a ranking algorithm might consider when determining the relevance and importance of a web page for a specific search query. As search engines continually update and improve their ranking algorithms, it's important for companies to stay up-to-date with best practices for search engine optimization (SEO) to maintain their visibility in search results.

Analysis study:

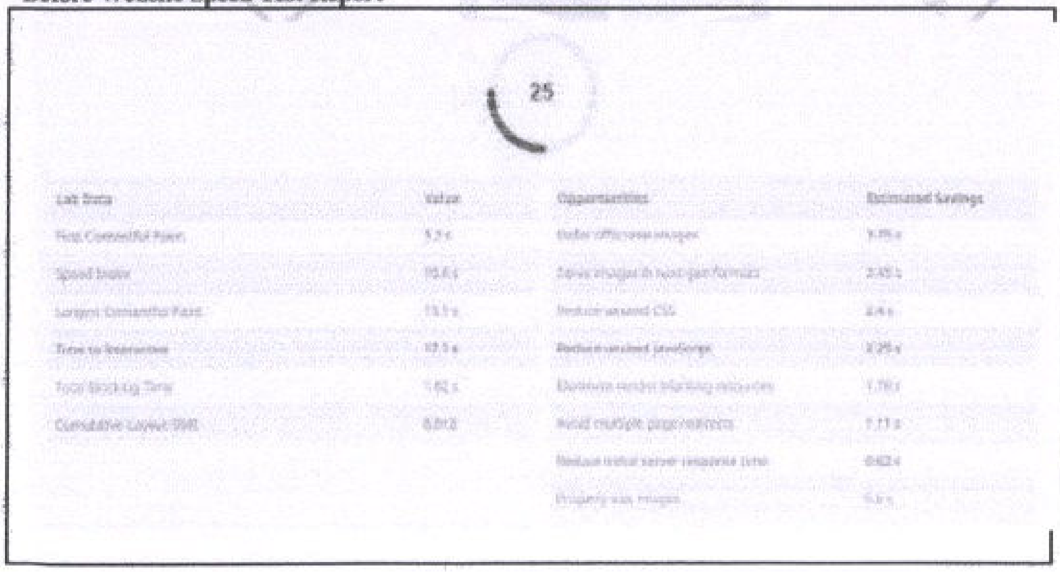
Before SEO (We find out this mistakes)

1. Missing Title tag
2. Missing Meta Description
3. Not Setted proper tracking
4. Alt text missing
5. Less Images
6. Low quality content
7. No internal linking
8. 10 Backlink
9. SSL Missing
10. Legal Pages Missing
11. Website hosted on bad server

Before Google Search Console Report



Before Website Speed Test Report

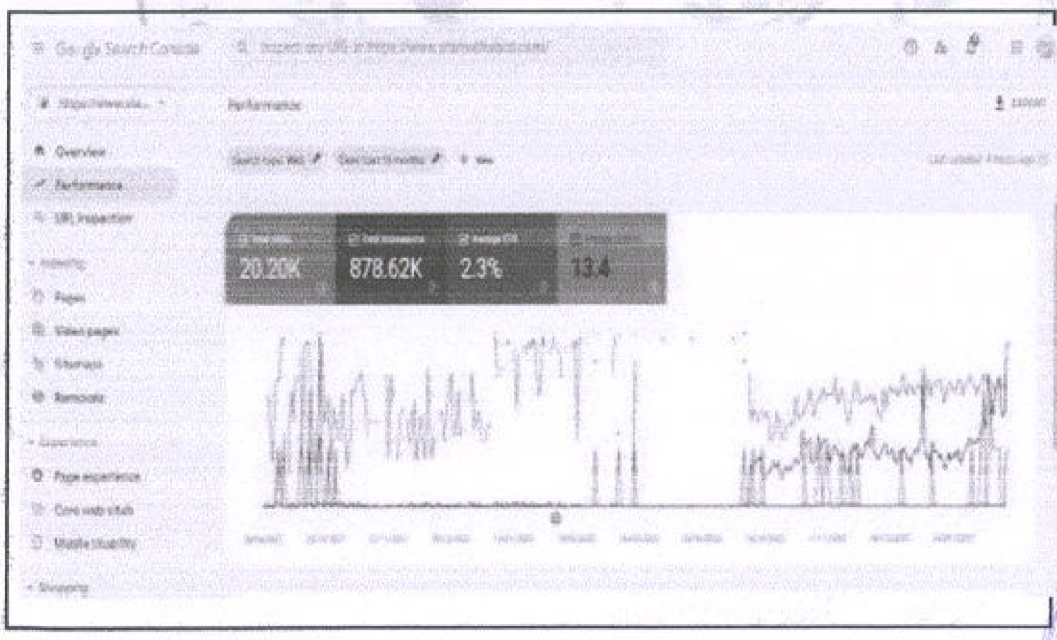


After The SEO:

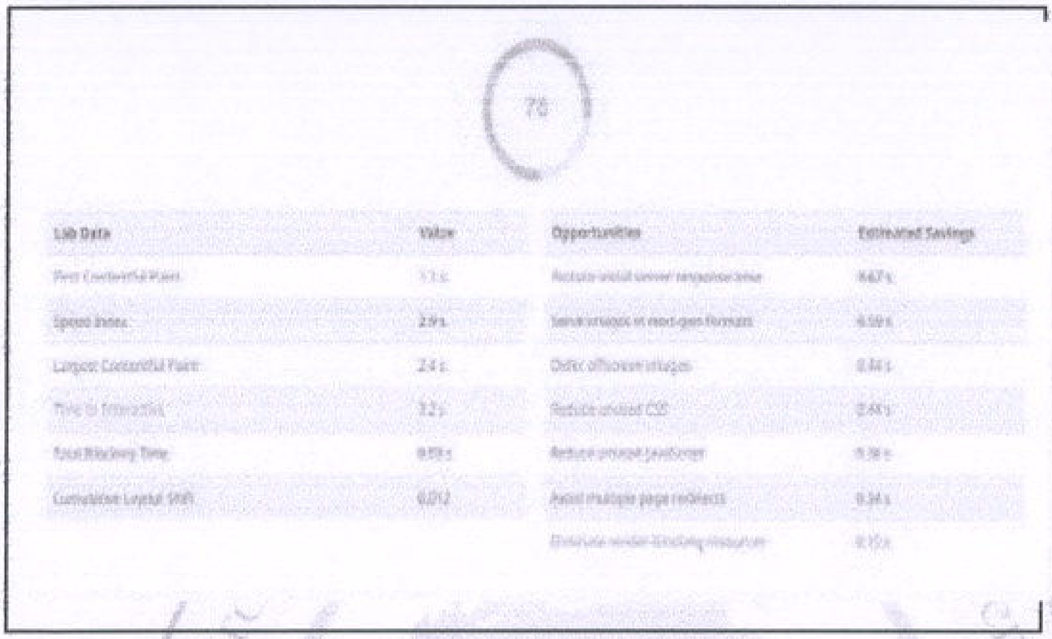
1. We restructure complete website
2. We write 20 SEO optimised titles tag
3. We Write 25+ New Meta Description
4. Keyword research. We use low competition keywords
5. We use short paragraph content
6. We add Images, info graphics and images on our blog pages
7. We set up proper internal linking structure
8. Newly setup Google Analytics and Search console
9. We share content on all social media platforms
10. We add new free stock images
11. We write 20 new blog post
12. We run Off-page SEO campaigns
13. Website caching setup
14. We Setup CDN and SSL
15. We newly add all legal pages like privacy policy T&C pages

After everything done we wait for results and finally we got impressive result

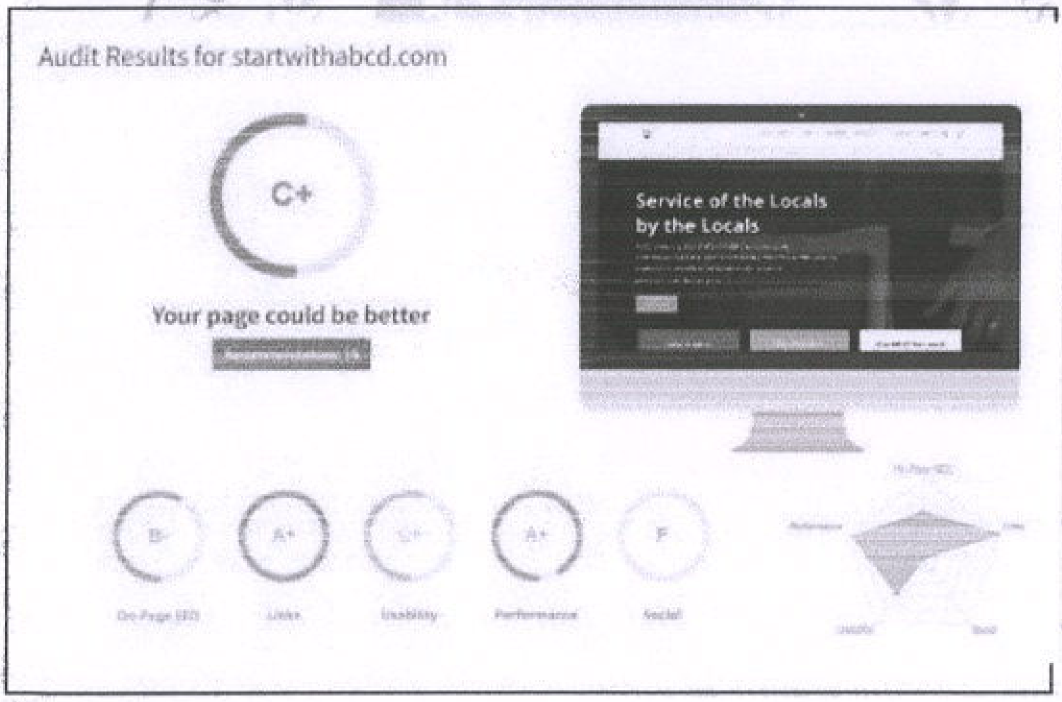
Before Google Search Console Report:



Before Website Speed Test Report :



Overall SEO:



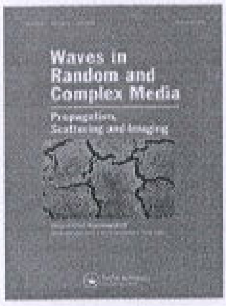
Conclusion:

In conclusion, a comprehensive SEO report should evaluate a website's performance in search engines and make recommendations for improvement. The report should cover a variety of factors, including keyword analysis, on-page optimization, backlink profile, technical SEO, and mobile optimization. The report should also use tools such as Google Analytics, Google Search

Console, and Google Trends to analyze website data and track progress. The final report should summarize the findings and make clear, actionable recommendations for improving the website's visibility and ranking in search results. It is important to regularly monitor and update the website's SEO strategies to stay ahead of changes in search algorithms and maintain a strong online presence.

References:

1. Carson, J. (2013). The Future of SEO. In J. Carson, The Future of SEO.
2. Search Engine Optimization (SEO): An hour a day. In J. G. Couzin, Search Engine Optimization (SEO): (2008). An hour a day.
3. Cutts, M. (2015). The Science of SEO: How Search Engines Work. In M. Cutts, The Science of SEO: How Search Engines Work.
4. Enge, E. (2016). The Art of SEO: How to Optimize Your Website for Search Engines. In E. Enge, The Art of SEO: How to Optimize Your Website for Search Engines.
5. Fishkin, R. (2017). SEO in the Digital Age: The Latest Trends and Best Practices. In R. Fishkin, SEO in the Digital Age: The Latest Trends and Best Practices.
6. James, D. SEO: A Comprehensive Guide. In SEO: A Comprehensive Guide" (p. 2012). 7. Johns, N. (2014). The Importance of Local SEO. In N. Johns, The Importance of Local SEO". 8. Kent, P. (2009). SEO for Dummies. In SEO for Dummies.
7. Munsell, J. (2010). In SEO: An Overview.
8. Nurmi, J. Moz - The Beginner's Guide to SEO. In Moz - The Beginner's Guide to SEO. Zillow. 11. White, M. (2011). The Impact of SEO on Business. In M. White, The Impact of SEO on Business.



Taylor & Francis
Taylor & Francis Group

Waves in Random and Complex Media

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/twrm20

On stability analysis of a fractional volterra integro delay differential equation in the context of Mittag–Leffler kernel

Aziz Khan, Pallavi S. Scindia, Sagar T. Sutar & Thabet Abdeljawad

To cite this article: Aziz Khan, Pallavi S. Scindia, Sagar T. Sutar & Thabet Abdeljawad (2023): On stability analysis of a fractional volterra integro delay differential equation in the context of Mittag–Leffler kernel, Waves in Random and Complex Media, DOI: 10.1080/17455030.2023.2223720

To link to this article: https://doi.org/10.1080/17455030.2023.2223720

Published online: 14 Jun 2023.

WAVES IN RANDOM AND COMPLEX MEDIA
https://doi.org/10.1080/17455030.2023.2223720

On stability analysis of a fractional volterra integro delay differential equation in the context of Mittag–Leffler kernel

Aziz Khan[a], Pallavi S. Scindia[b], Sagar T. Sutar[c] and Thabet Abdeljawad[a,d]

[a]Department of Mathematics and Sciences, Prince Sultan University, Riyadh, Saudi Arabia; [b]Department of Mathematics, College of Engineering Pune, Pune, Maharashtra, India; [c]Department of Mathematics, Dattajirao Kadam Arts, Science and Commerce college, Ichalkaranji, Maharashtra, India; [d]Department of Medical Research, China Medical University, Taichung, Taiwan

ABSTRACT
This paper deals with a nonlinear multi-derivative fractional Volterra integro-differential equation with finite delay involving a non-singular Mittag–Leffler kernel. Krasnoselskii's fixed-point theorem is used to obtain the current result. Also, we establish the generalization of Pachapate's inequality and provide its applications for the existence, uniqueness, stability, and data dependence of the suggested problem. All the results are justified by an example.

ARTICLE HISTORY
Received 5 December 2022
Accepted 5 June 2023

KEYWORDS
Integro-differential equations; data dependence results; Mittag–Leffler kernel; Ulam–Hyers–Rassias stability

MATHEMATICS SUBJECT CLASSIFICATIONS
26A33; 34A08; 34A12; 34A40

1. Introduction

Basic theoretical development and applications due to fractional differential operators, viz., Caputo and Riemann–Liouville involving singular kernels, have been found in [1, 2]. The arbitrary-order derivatives in the form of a non-singular kernel give a new direction to mathematical modeling. Caputo-Fabrizio fractional derivative [3] involves a non-singular kernel as an exponential, while Atangana–Baleanu fractional derivative [4] involves a non-singular Mittag–Leffler kernel. To know the details about modeling with these derivatives, one can refer to [5–11] and overall, the fractional order modeling is a strong tool that provides a unique perspective for understanding and modeling complicated systems with non-integer behavior and also references cited therein [12–18]. The development of the theory of FDEs with these derivatives for various classes has been studied in [19]. Kucche et al. [20] derived results for nonlinear FDEs using a non-singular Mittag–Leffler kernel to get a local, global, and extremal solution for nonlinear FDEs and nonlinear Hybrid FDEs.

The theory of Ulam-type stability was obtained due to a question raised by S.M.Ulam [21]. Its generalization in different directions results in various forms of Ulam-type stabilities. The study of such type of stability generates a variety of classes of differential equations using different techniques as found in [22, 23]. Pachpatte's inequality and its generalization play an important role in the study of integro-differential equations of integer order see [24].

CONTACT Thabet Abdeljawad tabdeljawad@psu.edu.sa

Utilizing the Caputo fractional differential operator, Kucche et al. [25] studied the problem for validity, convergence and error estimation due to Picard's method and Mohamed et al. [26] studied the problem for existence, uniqueness and numerical solution through different methods. Sutar et al. [27] studied multi-order differential equations involving Atangan–Baleanu–Riemann (ABR) type fractional derivative. Inspired by this work, in this paper, we consider a nonlinear multi-derivative fractional Volterra integro-differential equation with delay:

$$\frac{d}{dx}\eta(x) + {}^{ABR}{}_0\mathcal{D}_x^{\varrho}\eta(x) = \mathcal{U}\left(x, \eta_x, \int_0^x \mathcal{V}(x, y, \eta_y)\, dy\right), \quad x \in I = [0, X],\ X > 0, \tag{1}$$

$$\eta(x) = \varsigma(x) \in [-r, 0], \tag{2}$$

where ${}^{ABR}{}_0\mathcal{D}_x^{\varrho}$ denotes left ABR-type differential operator fractional order of order $\varrho, (0 < \varrho < 1)$ with a lower terminal 0. $\mathcal{U} : I \times C([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{V} : I \times I \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ are nonlinear continuous functions.

The class of equations that we have considered is to more generalize than that of studied and cover all the results obtained in [27] as a particular case.

In the Preliminary section, we recall basic definitions and results related to our work. In Section 3, we establish generalization of Pachpatte-type inequality which plays a key role to obtain our results. In Section 4, we obtain the necessary condition for the solution to exist and be unique. In Section 5, we derive estimates on the solution and various data dependence results. Section 6 deals with the Ulam-type stability results. The Krasnoselskii's fixed-point theorem and derived Pachpattes inequality play a key role in obtaining our results.

2. Preliminaries

Let $\mathbb{R}$ be the real Banach space with norm $|\cdot|$. Consider the Banach spaces $\mathcal{C} = C([-r, 0], \mathbb{R})$ and $\mathcal{B} = C([-r, X], \mathbb{R}), X > 0$ of all continuous functions defined on respective domains with respective suprimum norms $\|\cdot\|_{\mathcal{C}}$ and $\|\cdot\|_{\mathcal{B}}$. For any function $\eta \in \mathcal{B}$ and $x \in I$, η_x denotes the function in $\mathcal{C}$ and is defined by $\eta_x(\theta) = \eta(x + \theta)$.

Definition 2.1 ([4]): Suppose $\varrho \in [1, \infty)$ and $\mathcal{G}$ be an open subset of $\mathbb{R}$ then the Sobolev space $H^{\varrho}(\mathcal{G})$ is given by

$$H^{\varrho}(\mathcal{G}) = \left\{ f \in \mathcal{L}^2(\mathcal{G}) : D^{\beta} f \in \mathcal{L}^2(\mathcal{G}), \text{ for all } |\beta| \le \varrho \right\}.$$

Definition 2.2 ([4]): Let $\eta \in H^1(0, 1)$ and $0 < \varrho < 1$, the left ABR fractional derivative is given by

$${}^{ABR}{}_0\mathcal{D}_x^{\varrho}\eta(x) = \frac{B(\varrho)}{1-\varrho}\frac{d}{dx}\int_0^x \mathbb{E}_{\varrho}\left[-\frac{\varrho}{1-\varrho}(x-y)^{\varrho}\right]\eta(y)\, dy,$$

where $B(\varrho) > 0$ denotes a normalization function with property $B(0) = 1 = B(1)$ and $\mathbb{E}_{\varrho}$ is the Mittag–Leffler function.

Definition 2.3 ([28, 29]): The generalized Mittag–Leffler function $\mathbb{E}^{\gamma}_{\varrho,\beta}(z)$ for the complex ϱ, β, γ with $\mathrm{Re}(\varrho) > 0$ is defined as

$$\mathbb{E}^{\gamma}_{\varrho,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\varrho k + \beta)} \frac{z^k}{k!},$$

where $(\gamma)_k$ is the Pochhammer symbol given by

$$(\gamma)_0 = 1, \quad (\gamma)_k = \gamma(\gamma+1)\cdots(\gamma+k-1), \quad k = 1, 2, \ldots$$

Definition 2.4 ([28, 29]): For $\rho, \mu, \omega, \gamma \in \mathbb{C}$ $(Re(\rho), Re(\mu) > 0)$, we have fractional integral operator $\mathcal{E}^{\gamma}_{\rho,\mu,\omega;0+}$ on class $C([a,b])$ defined by

$$(\mathcal{E}^{\gamma}_{\rho,\mu,\omega;a+}\phi)(\tau) = \int_a^{\tau} (\tau - \sigma)^{\gamma-1} \mathbb{E}^{\gamma}_{\rho,\mu}[\omega(\tau - \sigma)^{\varrho}]\phi(\tau)\, \mathrm{d}\tau, \quad \tau \in [a, b].$$

Lemma 2.1 ([23]): *Let $0 < \varrho < 1$. Define the function Θ on $C(I)$ by*

$$(\Theta\omega)(\tau) = \frac{B(\varrho)}{1-\varrho}\left(\mathcal{E}^{1}_{\varrho,1,\frac{-\varrho}{1-\varrho};0+}\omega\right)(\tau), \omega \in C(I), \quad \tau \in J.$$

Then:

(i) *Θ is the bounded linear operator on $C(I)$.*
(ii) *Θ satisfies the Lipschitz condition.*
(iii) *$\Theta(S)$ is equicontinuous, where S is any bounded subset of $C(I)$.*
(iv) *Θ is invertible and for any $f \in C(I)$, the operator equation $\Theta\omega = f$ has a unique solution in $C(I)$.*

Lemma 2.2 (Krasnoselskii's fixed point theorem, [30]): *Let Ω be a Banach space. Let S be a bounded, closed, convex subset of Ω and let $\mathcal{F}_1, \mathcal{F}_2$ be maps of S into Ω such that $\mathcal{F}_1\omega + \mathcal{F}_2\eta \in S$ for every pair $\omega, \eta \in S$. If $\mathcal{F}_1$ is the contraction and $\mathcal{F}_2$ is continuous, then the equation*

$$\mathcal{F}_1\omega + \mathcal{F}_2\omega = \omega$$

has a solution on S.

Lemma 2.3 ([23]): *Let $h \in C(I)$ be any function, the function $\eta \in C(I)$ is a solution of problem*

$$\frac{\mathrm{d}\eta}{\mathrm{d}x} + {}^{ABR}{}_0\mathcal{D}^{\varrho}_x\eta(x) = h(x), x \in I,$$
$$\eta(0) = \eta_0 \in \mathbb{R},$$

then η is a solution

$$\eta(x) = \eta_0 - \frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_{\varrho}\left[\frac{-\varrho}{1-\varrho}(x-y)^{\varrho}\right]\eta(y)\,\mathrm{d}y + \int_0^x h(y)\,\mathrm{d}y, \quad x \in I.$$

3. Generalization of inequality:

We have modified the following inequality.

Lemma 3.1 ([31], Theorem 1.7.4/(iii) on page 35): *Let u, f, g and h be nonnegative continuous functions defined on $\mathbb{R}^+$ and u_0 be a nonnegative constant. If*

$$u(x) \leq u_0 + \int_0^x f(y)u(y)\,dy + \int_0^x g(y)\left(u(y) + \int_0^y h(z)u(z)\,dz\right) dy, x \in \mathbb{R}^+,$$

then for any $x \in \mathbb{R}^+$,

$$u(x) \leq u_0\left[\exp\left(\int_0^x f(z)\,dz\right) + \int_0^x g(y)\exp\left(\int_0^y [f(z)+g(z)+h(z)]\,dz\right) \times \exp\left(\int_y^x f(z)\,dz\right) dy\right].$$

To establish uniqueness, data dependence and Ulam–Hyers stability results for our problem, we show that the following theorem is the new generalization of the generalized Pachpatte's inequality given in Lemma 3.1. Its proof is close to the proof of Theorem 1.7.4 ([31], page 39) and can be completed on a similar line.

Theorem 3.2: *Let u, f, g and h be nonnegative continuous functions defined on $\mathbb{R}^+$ and n(x) is a positive and nondecreasing continuous function defined on $\mathbb{R}^+$. If*

$$u(x) \leq n(x) + \int_0^x f(y)u(y)\,dy + \int_0^x g(y)\left(u(y) + \int_0^y h(z)u(z)\,dz\right) dy, x \in \mathbb{R}^+, \quad (3)$$

then for any $x \in \mathbb{R}^+$,

$$u(x) \leq n(x)\left[\exp\left(\int_0^x f(z)\,dz\right) + \int_0^x g(y)\exp\left(\int_0^y [f(z)+g(z)+h(z)]\,dz\right) \times \exp\left(\int_y^x f(z)\,dz\right) dy\right]. \quad (4)$$

Proof: For any $x \in \mathbb{R}^+$, using the characteristics of $n(x)$, we write from inequality (3)

$$u(x) \leq n(x) + \int_0^x f(y)\frac{u(y)}{n(y)}n(x)\,dy + \int_0^x g(y)\left(\frac{u(y)}{n(y)}n(x) + \int_0^y h(z)\frac{u(z)}{n(z)}n(x)\,dz\right) dy$$
$$= n(x)\left[1 + \int_0^x f(y)\frac{u(y)}{n(y)}\,dy + \int_0^x g(y)\left(\frac{u(y)}{n(y)} + \int_0^y h(z)\frac{u(z)}{n(z)}\,dz\right) dy\right].$$

This gives

$$\frac{u(x)}{n(x)} \leq 1 + \int_0^x f(y)\frac{u(y)}{n(y)}\,dy + \int_0^x g(y)\left(\frac{u(y)}{n(y)} + \int_0^y h(z)\frac{u(z)}{n(z)}\,dz\right) dy.$$

As an application of the Lemma 3.1 to the above inequality with $u(x) = \frac{u(x)}{n(x)}$ and $u_0 = 1$, we get the desired inequality (4).

4. Existence of solutions

Theorem 4.1: *The function $\eta \in \mathcal{B}$ is a solution of problem* (30)–(31) *if and only if η is a solution of integral equation*

$$\eta(x) = \varsigma(0) - \frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]\eta(y)\,dy$$
$$+\int_0^x \mathcal{U}\left(y,\eta_y,\int_0^y \mathcal{V}(y,z,\eta_z)\,dz\right)dy, \quad x\in I, \tag{5}$$
$$\eta(x) = \varsigma(x), \quad x\in[-r,0]. \tag{6}$$

Proof: The proof follows by taking $h(x) = \mathcal{U}(x,\eta_x,\int_0^x \mathcal{V}(x,y,\eta_y)\,dy), x\in I$, and using the initial condition $\eta(0)=\varsigma(0)$ in the Lemma 2.3. ■

We need the following assumption to obtain our results:

(A1) There are functions $L_\mathcal{U}, L_\mathcal{V} \in C(J,\mathbb{R}_+)$ such that
 (i) $|\mathcal{U}(x,\zeta_1,\eta_1) - \mathcal{U}(x,\zeta_2,\eta_2)| \le L_\mathcal{U}(x)(|\zeta_1-\zeta_2|_\mathcal{C} + |\eta_1-\eta_2|)$,
 (ii) $|\mathcal{V}(x,y,\zeta_1) - \mathcal{V}(x,y,\zeta_2)| \le L_\mathcal{V}(y)\,|\zeta_1-\zeta_2|_\mathcal{C}$,
 for all $x,y\in J$, $\zeta_1,\zeta_2\in\mathcal{C}$ and $\eta_1,\eta_2\in\mathcal{B}$.

Theorem 4.2 (Existence Results): *Under the hypothesis [A_1], if $0 < L_1 < \zeta$, where $\zeta = $* $\min\{1, \frac{1}{2X+X^2(L_2+M_\mathcal{V})}\}$, *then the problem* (30)–(31) *has a solution in $\mathcal{B}$ provided*

$$\frac{2B(\varrho)X\mathbb{E}_{\varrho,2}\left(\frac{\varrho}{1-\varrho}X^\varrho\right)}{1-\varrho} < 1, \tag{7}$$

where $L_1 = \sup_{x\in I} L_\mathcal{U}(x), L_2 = \sup_{x\in I} L_\mathcal{V}(x), M_\mathcal{U} = \sup_{x\in I}\mathcal{U}(x,0,0)$ *and* $M_\mathcal{V} = \sup_{x,z\in J}\mathcal{V}(x,z,0)$

Proof: Define

$$R = \frac{|\omega_0| + M_\mathcal{U}X}{1 - L_1\left(X+\frac{X^2}{2}(L_2+M_\mathcal{V})\right) - \frac{B(\varrho)X\mathbb{E}_{\varrho,2}\left(\frac{\varrho}{1-\varrho}X^\varrho\right)}{1-\varrho}}.$$

Therefore, choosing L_1,ζ and condition (7), we have $R>0$.
Consider the set,

$$\mathcal{S} = \{\eta\in\mathcal{B}: \ \|\eta\| \le R\}.$$

One can verify that $\mathcal{S}$ is closed, convex and bounded subset of Banach space $\mathcal{B}$. Consider the operators $\mathcal{F}_1:\mathcal{S}\to\mathcal{B}$ and $\mathcal{F}_2:\mathcal{S}\to\mathcal{B}$ defined by

$$(\mathcal{F}_1\eta)(x) = \begin{cases}\varsigma(0) + \int_0^x \mathcal{U}\left(y,\eta_y,\int_0^y\mathcal{V}(y,z,\eta_z)\,dz\right)dy, & x\in I,\\ \varsigma(x), & x\in[-r,0],\end{cases}$$

and

$$(\mathcal{F}_2\eta)(x) = \begin{cases} -(\mathcal{F}\eta)(x), \ x \in I, \\ 0, \ x \in [-r, 0], \end{cases}$$

where consider $\mathcal{F}$ that is given in Lemma 2.1. The equivalent fractional integral Equations (5)–(6) to the problem (30)–(31) can be written as operator equations in the following form

$$\eta = \mathcal{F}_1\eta + \mathcal{F}_2\eta, \quad \eta \in \mathcal{B}.$$

We show the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ complete all the conditions of Lemma 2.2. The proof of the same has been given in the following steps.

Step (1) $\mathcal{F}_1$ is the contraction.

Let $\eta,\omega \in \mathcal{B}$. Using the definition of $\mathcal{F}_1$, we have for any $x \in [-r, 0]$,

$$|(\mathcal{F}_1\eta)(x) - (\mathcal{F}_1\omega)(x)| = |\varsigma(x) - \varsigma(x)| = 0. \tag{8}$$

Again for any $x \in I$ we obtain using hypothesis (A1)

$$\begin{aligned}
&|(\mathcal{F}_1\eta)(x) - (\mathcal{F}_1\omega)(x)| \\
&\le \int_0^x \left| \mathcal{U}\left(y, \eta_y, \int_0^y \mathcal{V}(y, z, \eta_z)\, dz\right) dy - \mathcal{U}\left(y, \omega_y, \int_0^y \mathcal{V}(y, z, \omega_z)\, dz\right) \right| dy \\
&\le \int_0^x L_{\mathcal{U}}(y) \left(|\eta_y - \omega_y|_{\mathcal{C}} + \int_0^y |\mathcal{V}(y, z, \eta_z)\, dz - \mathcal{V}(y, z, \omega_z)|\, dz \right) dy \\
&\le \int_0^x L_{\mathcal{U}}(y) \left(|\eta_y - \omega_y|_{\mathcal{C}} + \int_0^y L_{\mathcal{V}}(z) |\eta_z - \omega_z|_{\mathcal{C}}\, dz \right) dy \\
&\le \int_0^x L_1 \left(|\eta_y - \omega_y|_{\mathcal{C}} + L_2 \int_0^y |\eta_z - \omega_z|_{\mathcal{C}}\, dz \right) dy \\
&\le \|\eta - \omega\|_{\mathcal{B}} \left(L_1 x + L_1 L_2 \frac{x^2}{2} \right) \le \left(L_1 X + L_1 L_2 \frac{X^2}{2} \right) \|\eta - \omega\|_{\mathcal{B}} \\
&\le \frac{L_1}{2} (2X + L_2 X^2) \|\eta - \omega\|_{\mathcal{B}} \le \frac{L_1}{2} (2X + L_2 [X^2 + M_{\mathcal{V}}]) \|\eta - \omega\|_{\mathcal{B}} \\
&\le \frac{L_1}{2\zeta} \|\eta - \omega\| \le \frac{1}{2} \|\eta - \omega\|_{\mathcal{B}}.
\end{aligned}$$

This gives

$$|(\mathcal{F}_1\eta)(x) - (\mathcal{F}_1\omega)(x)| \le \|\omega - \eta\|_{\mathcal{B}}, \ x \in I, \ \omega, \eta \in \mathcal{B}. \tag{9}$$

Using inequalities (8) and (9), we get

$$\|\mathcal{F}_1\eta - \mathcal{F}_1\omega\|_{\mathcal{B}} \le \|\omega - \eta\|_{\mathcal{B}}, \quad \omega, \eta \in \mathcal{B}.$$

This shows that $\mathcal{F}_1$ is a contraction.

Step (2) $\mathcal{F}_2$ is continuous.

applying Ascoli-Arzela theorem and Lemma 2.1, one can easily prove that the operator $\mathcal{F}_2 = -\mathcal{F}$ is continuous.

Step (3) $\mathcal{F}_1\omega + \mathcal{F}_2\eta \in \mathcal{S}$, for $\omega, \eta \in \mathcal{S}$.
For any $\omega, \eta \in \mathcal{S}$, using Theorem 2.1, we obtain

$$\begin{aligned}
|(\mathcal{F}_1\omega + \mathcal{F}_2\eta)(x)| &\le |(\mathcal{F}_1\omega)(x)| + |(\mathcal{F}_2\eta)(x)| \\
&\le |\omega_0| + \int_0^x \left|\mathcal{U}\left(y, \eta_y, \int_0^y \mathcal{V}(y,z,\eta_z)\,dz\right)\right| dy + \frac{B(\varrho)}{1-\varrho} X\mathbb{E}_{\varrho,2}\left(\frac{\varrho}{1-\varrho}X^{\varrho}\right)\|\eta\| \\
&\le |\omega_0| + \int_0^x \left|\mathcal{U}\left(y, \eta_y, \int_0^y \mathcal{V}(y,z,\eta_z)\,dz\right) - \mathcal{U}(y,0,0)\right| dy + \int_0^x |\mathcal{U}(y,0,0)|\,dy \\
&\quad + \frac{B(\varrho)}{1-\varrho} X\mathbb{E}_{\varrho,2}\left(\frac{\varrho}{1-\varrho}X^{\varrho}\right) R \\
&\le |\omega_0| + \int_0^x L_{\mathcal{U}}(y)\left(|\eta_y|_C + \int_0^y |\mathcal{V}(y,z,\eta_z)|\,dz\right) dy + M_{\mathcal{U}}x \\
&\quad + \frac{B(\varrho)}{1-\varrho} X\mathbb{E}_{\varrho,2}\left(\frac{\varrho}{1-\varrho}X^{\varrho}\right) R \\
&\le |\omega_0| + L_1 \int_0^x \left(|\eta_y|_C + \int_0^y |\mathcal{V}(y,z,\eta_z) - \mathcal{V}(y,z,0)|\,dz + \int_0^y |\mathcal{V}(y,z,0)|\,dz\right) dy \\
&\quad + M_{\mathcal{U}}x + \frac{B(\varrho)}{1-\varrho} X\mathbb{E}_{\varrho,2}\left(\frac{\varrho}{1-\varrho}X^{\varrho}\right) R \\
&\le |\omega_0| + L_1 \int_0^x \left(\|\eta\| + \int_0^y L_{\mathcal{V}}(z)\,|\eta_z|_C\,dz + M_{\mathcal{V}} \int_0^y |\eta_z|_C\,dz\right) dy \\
&\quad + M_{\mathcal{U}}x + \frac{B(\varrho)}{1-\varrho} X\mathbb{E}_{\varrho,2}\left(\frac{\varrho}{1-\varrho}X^{\varrho}\right) R \\
&\le |\omega_0| + L_1\|\eta\| \int_0^x (1 + L_2 y + M_{\mathcal{V}}y)\,dy + M_{\mathcal{U}}x + \frac{B(\varrho)}{1-\varrho} X\mathbb{E}_{\varrho,2}\left(\frac{\varrho}{1-\varrho}X^{\varrho}\right) R \\
&\le |\omega_0| + L_1 R\left(y + (L_2 + M_{\mathcal{V}})\frac{x^2}{2}\right) + M_{\mathcal{U}}x + \frac{B(\varrho)}{1-\varrho} X\mathbb{E}_{\varrho,2}\left(\frac{\varrho}{1-\varrho}X^{\varrho}\right) R \\
&\le |\omega_0| + L_1 R\left(X + (L_2 + M_{\mathcal{V}})\frac{X^2}{2}\right) + M_{\mathcal{U}}X + \frac{B(\varrho)}{1-\varrho} X\mathbb{E}_{\varrho,2}\left(\frac{\varrho}{1-\varrho}X^{\varrho}\right) R,
\end{aligned} \tag{10}$$

with the help of definition of R, i.e. condition(7), we obtained the following

$$|\omega_0| + M_{\mathcal{U}}X = R\left(1 - L_1\left(X + \frac{X^2}{2}(L_2 + M_{\mathcal{V}})\right) - \frac{B(\varrho)X\mathbb{E}_{\varrho,2}\left(\frac{\varrho}{1-\varrho}X^{\varrho}\right)}{1-\varrho}\right). \tag{11}$$

We write from inequalities (10) and (11)

$$|(\mathcal{F}_1\omega + \mathcal{F}_2\eta)(x)| \le R, \quad x \in I.$$

This gives

$$\|\mathcal{F}_1\omega + \mathcal{F}_2\eta\|_{\mathcal{B}} \le R, \text{ for all } \omega, \quad \eta \in \mathcal{S}.$$

This shows that $\mathcal{F}_1\omega + \mathcal{F}_2\eta \in \mathcal{S}$, for $\omega, \eta \in \mathcal{S}$.

From steps 1 to 3, it follows that all the conditions of Lemma 2.2 are satisfied. Therefore by applying it, the operator equation

$$\omega = \mathcal{F}_1\omega + \mathcal{F}_2\omega,$$

has a fixed point in $\mathcal{S}$, which is a solution to the problem (30)–(31). This completes the proof of the theorem. ■

Note that Krasnoselskii's fixed point theorem does not guarantee the uniqueness of the solution, but in the following result we have shown that with the same assumptions of existence result, we have uniqueness result.

Theorem 4.3 (Uniqueness): *Considering the assumptions of Theorem 4.2, the problem* (30)–(31) *has the unique solution in* $\mathcal{B}$.

Proof: Let $\eta, \xi \in \mathcal{B}$ be two solutions of the problem (30)–(31). We find for any $x \in I$

$$\begin{aligned}
&|\eta(x) - \xi(x)| \\
&= \Bigg|\Bigg\{\varsigma(0) - \frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]\eta(y)\,dy \\
&\quad + \int_0^x \mathcal{U}\left(y, \eta_y, \int_0^y \mathcal{V}(y,z,\eta_z)\,dz\right)dy\Bigg\} \\
&\quad - \Bigg\{\varsigma(0) - \frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]\xi(y)\,dy \\
&\quad + \int_0^x \mathcal{U}\left(y, \xi_y, \int_0^y \mathcal{V}(y,z,\xi_z)\,dz\right)dy\Bigg\}\Bigg| \\
&\le \frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_\varrho\left[\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]|\eta(y) - \xi(y)|\,dy \\
&\quad + \int_0^x L_{\mathcal{U}}(y)\left(|\eta_y - \xi_y|_{\mathcal{C}} + \int_0^y L_{\mathcal{V}}(z)|\eta_z - \xi_z|_{\mathcal{C}}\,dz\right)dy.
\end{aligned}$$

Consider the function γ defined by

$$\gamma(x) = \sup_{-r<t<x} |(\eta - \xi)(t)|, \quad x \in I, \tag{12}$$

then $|(\eta - \xi)(t)|_{\mathcal{C}} \le \gamma(x)$, $\forall\, x \in I$ and there is $x^* \in [-r, x]$ such that $\gamma(x) = |(\eta - \xi)(x^*)|$. Hence, for $x^* \in [0, x]$ we have

$$\begin{aligned}
\gamma(x) &\le \frac{B(\varrho)}{1-\varrho}\int_0^{x^*} \mathbb{E}_\varrho\left[\frac{\varrho}{1-\varrho}(x^* - y)^\varrho\right]\gamma(y)\,dy \\
&\quad + \int_0^{x^*} L_{\mathcal{U}}(y)\left(\gamma(y) + \int_0^y L_{\mathcal{V}}(z)\gamma(z)\,dz\right)dy \\
&\le \frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_\varrho\left[\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]\gamma(y)\,dy
\end{aligned}$$

$$+\int_0^x L_{\mathcal{U}}(y)\left(\gamma(y)+\int_0^y L_{\mathcal{V}}(z)\gamma(z)\,dz\right)dy.$$

Applying the integral inequality from Theorem 3.2, with

$$n(t)=0,\quad f(y)=\frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left[\frac{\varrho}{1-\varrho}(x-y)^\varrho\right],\quad g(y)=L_{\mathcal{U}}(y)\quad\text{and}\quad h(y)=L_{\mathcal{V}}(y),$$

we get

$$|\eta(x)-\xi(x)|\le 0,\quad x\in I,$$

which verify the uniqueness of the solution of the problem (30)–(31). ■

5. Estimate on solution and data dependence

Theorem 5.1: *Let's suppose assumptions of Theorem 4.2, if $\eta(x)$ is a solution of problem (30)–(31), then*

$$\begin{aligned}|\eta(x)|\le(|\varsigma(0)|+M_{\mathcal{U}}X)&\left[\exp\left(\int_0^x\left\{\frac{B(\varrho)}{1-\varrho}\mathbb{E}_\varrho\left[\frac{\varrho}{1-\varrho}(X-y)^\varrho\right]+L_{\mathcal{U}}(y)\right\}\right)dy\right.\\&+\int_0^x L_{\mathcal{U}}(y)\exp\left(\int_0^y\left[\frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left(\frac{\varrho}{1-\varrho}(X-z)^\varrho\right)+L_{\mathcal{U}}(z)+L_{\mathcal{V}}(z)\right]dz\right)\\&\left.\times\exp\left(\int_y^X\frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]dz\right)dy\right].\end{aligned}$$

Proof: If $\eta(x)$ is a solution of problem (30)–(31), then it satisfies equivalent integral Equation (5)–(6). Hence, we write for any $x\in I$,

$$\begin{aligned}|\eta(x)|&\le|\varsigma(0)|+\frac{B(\varrho)}{1-\varrho}\int_0^x\mathbb{E}_\varrho\left[\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]|\eta(y)|\,dy\\&\quad+\int_0^x\left|\mathcal{U}\left(y,\eta_y,\int_0^y\mathcal{V}(y,z,\eta_z)\,dz\right)\right|dy\\&\le|\varsigma(0)|+\frac{B(\varrho)}{1-\varrho}\int_0^x\mathbb{E}_\varrho\left[\frac{\varrho}{1-\varrho}(X-y)^\varrho\right]|\eta(y)|\,dy\\&\quad+\int_0^x\left|\mathcal{U}\left(y,\eta_y,\int_0^y\mathcal{V}(y,z,\eta_z)\,dz\right)-\mathcal{U}(y,0,0)\right|dy+\int_0^x|\mathcal{U}(y,0,0)|\,dy\\&\le|\varsigma(0)|+\frac{B(\varrho)}{1-\varrho}\int_0^x\mathbb{E}_\varrho\left[\frac{\varrho}{1-\varrho}(X-y)^\varrho\right]|\eta(y)|\,dy\\&\quad+\int_0^x L_{\mathcal{U}}(y)\left(|\eta_y|_{\mathcal{C}}+\int_0^y|\mathcal{V}(y,z,\eta_z)|\,dz\right)dy+M_{\mathcal{U}}x\\&\le|\varsigma(0)|+M_{\mathcal{U}}X+\frac{B(\varrho)}{1-\varrho}\int_0^x\mathbb{E}_\varrho\left[\frac{\varrho}{1-\varrho}(X-y)^\varrho\right]|\eta(y)|\,dy\\&\quad+\int_0^x L_{\mathcal{U}}(y)\left(|\eta_y|_{\mathcal{C}}+\int_0^y|\mathcal{V}(y,z,\eta_z)-\mathcal{V}(y,z,0)|\,dz+\int_0^y|\mathcal{V}(y,z,0)|\,dz\right)dy\\&\le|\varsigma(0)|+M_{\mathcal{U}}X+\frac{B(\varrho)}{1-\varrho}\int_0^x\mathbb{E}_\varrho\left[\frac{\varrho}{1-\varrho}(X-y)^\varrho\right]|\eta(y)|\,dy\end{aligned}$$

$$+\int_0^x L_{\mathcal{U}}(y)\left(|\eta_y|_C+\int_0^y L_{\mathcal{U}}(z)\,|\eta_z|_C\,\mathrm{d}z+M_{\mathcal{V}}y\right)\mathrm{d}y$$

$$\leq|\varsigma(0)|+M_{\mathcal{U}}X+\int_0^x\frac{B(\varrho)}{1-\varrho}\mathbb{E}_\varrho\left[\frac{\varrho}{1-\varrho}(X-y)^\varrho\right]|\eta(y)|\,\mathrm{d}y$$

$$+\int_0^x L_{\mathcal{U}}(y)\left(|\eta(y)|+\int_0^y L_{\mathcal{U}}(z)\,|\eta(z)|\,\mathrm{d}z+M_{\mathcal{V}}y\right)\mathrm{d}y$$

$$\leq|\varsigma(0)|+M_{\mathcal{U}}X+\int_0^x\left\{\frac{B(\varrho)}{1-\varrho}\mathbb{E}_\varrho\left[\frac{\varrho}{1-\varrho}(X-y)^\varrho\right]+L_{\mathcal{U}}(y)\right\}|\eta(y)|\,\mathrm{d}y$$

$$+\int_0^x L_{\mathcal{U}}(y)\left(M_{\mathcal{V}}y+\int_0^y L_{\mathcal{U}}(z)\,|\eta(z)|\,\mathrm{d}z\right)\mathrm{d}y$$

Applying Theorem 3.2, we obtain

$$|\eta(x)|\leq(|\varsigma(0)|+M_{\mathcal{U}}X)\left[\exp\left(\int_0^x\left\{\frac{B(\varrho)}{1-\varrho}\mathbb{E}_\varrho\left[\frac{\varrho}{1-\varrho}(X-y)^\varrho\right]+L_{\mathcal{U}}(y)\right\}\right)\mathrm{d}y\right.$$
$$+\int_0^x L_{\mathcal{U}}(y)\exp\left(\int_0^y\left[\frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left(\frac{\varrho}{1-\varrho}(X-z)^\varrho\right)+L_{\mathcal{U}}(z)+L_{\mathcal{V}}(z)\right]\mathrm{d}z\right)$$
$$\left.\times\exp\left(\int_y^X\frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]\mathrm{d}z\right)\mathrm{d}y\right].$$

■

Theorem 5.2 (Dependence on initial conditions): *Assume that the functions $\mathcal{U}$ and $\mathcal{V}$ satisfy conditions* (A1). *Let η_i $(i=1,2)$ be the solutions to the following problem corresponding to $\varsigma_i(x)$ $(i=1,2)$,*

$$\frac{\mathrm{d}}{\mathrm{d}x}\eta(x)+{}^{ABR}{}_0\mathcal{D}_x^\varrho\eta(x)=\mathcal{U}\left(x,\eta_x,\int_0^x\mathcal{V}(x,y,\eta_y)\,\mathrm{d}y\right),\quad x\in I=[0,X],\ X>0,\tag{13}$$

$$\eta(x)=\varsigma_i(x),\quad(i=1,2),\in[-r,0].\tag{14}$$

Then

$$|\eta(x)-\xi(x)|$$
$$\leq|\varsigma_1-\varsigma_2|_C\left[\exp\left(\int_0^X\frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-z)^\varrho\right]\mathrm{d}z\right)\right.$$
$$+\int_0^X L_{\mathcal{U}}(y)\exp\left(\int_0^y\left[\frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left(-\frac{\varrho}{1-\varrho}(x-z)^\varrho\right)+L_{\mathcal{U}}(z)+L_{\mathcal{V}}(z)\right]\mathrm{d}z\right)$$
$$\left.\times\exp\left(\int_y^X\frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]\mathrm{d}z\right)\mathrm{d}y\right].\tag{15}$$

Proof: Let any $\varsigma_i(x)$ $(i=1,2)$, and η_i $(i=1,2)$ be the corresponding solutions to the problems (13)–(14). Then by the hypothesis (A1) we have

$$|\eta_1(x)-\eta_2(x)|$$

$$\leq |\varsigma_1(0) - \varsigma_2(0)| + \frac{B(\varrho)}{1-\varrho} \int_0^x \mathbb{E}_\varrho \left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho \right] |\eta_1(y) - \eta_2(y)| \, dy$$
$$+ \int_0^x L_{\mathcal{U}}(y) \left(|(\eta_1 - \eta_2)(y)|_C + \int_0^y L_{\mathcal{V}}(z) |(\eta_1 - \eta_2)(z)|_C \, dz \right) dy. \quad (16)$$

With the function γ in (12) we have

$$\gamma(x) \leq |\varsigma_1 - \varsigma_2|_C + \frac{B(\varrho)}{1-\varrho} \int_0^x \mathbb{E}_\varrho \left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho \right] \gamma(y) \, dy$$
$$+ \int_0^x L_{\mathcal{U}}(y) \left(\gamma(y) + \int_0^y L_{\mathcal{V}}(z) \gamma(z) \, dz \right) dy.$$

Applying the integral inequality given in Theorem 3.2, with

$$u_0 = |\varsigma_1 - \varsigma_2|_C, \; f(y) = \frac{B(\varrho)}{1-\varrho} \mathbb{E}_\varrho \left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho \right],$$
$$g(x) = L_{\mathcal{U}}(x) \quad \text{and} \quad h(x) = L_{\mathcal{V}}(x),$$

we get

$$\gamma(x) \leq |\varsigma_1 - \varsigma_2|_C \left[\exp\left(\int_0^x \frac{B(\varrho)}{1-\varrho} \mathbb{E}_\varrho \left[-\frac{\varrho}{1-\varrho}(x-z)^\varrho \right] dz \right) \right.$$
$$+ \int_0^x L_{\mathcal{U}}(y) \exp\left(\int_0^y \left[\frac{B(\varrho)}{1-\varrho} \mathbb{E}_\varrho \left(-\frac{\varrho}{1-\varrho}(x-z)^\varrho \right) + L_{\mathcal{U}}(z) + L_{\mathcal{V}}(z) \right] dz \right)$$
$$\left. \times \exp\left(\int_y^x \frac{B(\varrho)}{1-\varrho} \mathbb{E}_\varrho \left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho \right] dz \right) dy \right],$$

which gives the desired inequality (15). ■

To discuss the data dependence result, we consider the problem

$$\frac{d}{dx}\xi(x) + {}^{ABR}{}_0\mathcal{D}_\tau^\varrho \xi(x) = \bar{\mathcal{U}}\left(x, \xi_x, \int_0^x \bar{\mathcal{V}}(x, y, \xi_y) \, dy \right), x \in I = [0, X], \; X > 0, \quad (17)$$
$$\xi(x) = \bar{\varsigma}(x) \in [-r, 0], \quad (18)$$

Theorem 5.3 (Dependence on functions): *Let hypothesis* (A1) *holds. Suppose that*

(A2) *there are functions* $P, Q > 0$ *such that*
- (i) $|\mathcal{U}(x, \zeta_1, \eta_1) - v(y, \zeta_1, \eta_2)| \leq P$;
- (ii) $|\varsigma(x) - \bar{\varsigma}(x)| \leq Q$;

for all $x, y \in J$, $\zeta_1, \zeta_2 \in C$ *and* $\eta_1, \eta_2 \in \mathcal{X}$. *If* η *and* ξ *are the solutions of problem* (30)–(31) *and* (17)–(18), *respectively, then*

$$|\eta(x) - \xi(x)| \leq (Q + PX) \left[\exp\left(\int_0^X \frac{B(\varrho)}{1-\varrho} \mathbb{E}_\varrho \left[-\frac{\varrho}{1-\varrho}(x-z)^\varrho \right] dz \right) \right.$$
$$+ \int_0^X L_{\mathcal{U}}(y)$$

$$\times \exp\left(\int_0^y \left[\frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left(-\frac{\varrho}{1-\varrho}(x-z)^\varrho\right) + \mathcal{L}_{\mathcal{U}}(z) + \mathcal{L}_{\mathcal{V}}(z)\right] dz\right)$$
$$\times \exp\left(\int_y^x \frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right] dz\right) dy\Bigg]. \tag{19}$$

Proof: Since η, ξ are the solutions of Equations (30)–(31) and (17)–(18) receptively. We find for $x \in [0, X]$

$$\begin{aligned}
&|\eta(x) - \xi(x)| \\
&\leq |\varsigma(0) - \bar{\varsigma}(0)| + \frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]|\eta(y) - \xi(y)|\,dy \\
&\quad + \int_0^x \left|\mathcal{U}\left(y, \eta_y, \int_0^y \mathcal{V}(y,z,\eta_z)\,dz\right) - \bar{\mathcal{U}}\left(x, \xi_x, \int_0^x \bar{\mathcal{V}}(y,z,\xi_z)\,dz\right)\right| dy \\
&\leq |\varsigma(0) - \bar{\varsigma}(0)| + \frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]|\eta(y) - \xi(y)|\,dy \\
&\quad + \int_0^x \left|\mathcal{U}\left(y, \eta_y, \int_0^y \mathcal{V}(y,z,\eta_z)\,dz\right) - \mathcal{U}\left(x, \xi_x, \int_0^x \mathcal{V}(y,z,\xi_z)\,dz\right)\right| dy \\
&\quad + \int_0^x \left|\mathcal{U}\left(y, \eta_y, \int_0^y \mathcal{V}(y,z,\eta_z)\,dz\right) - \bar{\mathcal{U}}\left(x, \xi_x, \int_0^x \bar{\mathcal{V}}(y,z,\xi_z)\,dz\right)\right| dy \\
&\leq Q + \frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]|\eta(y) - \xi(y)|\,dy + \int_0^x \mathcal{L}_{\mathcal{U}}(y)\Big(|\eta_y - \xi_y|_C \\
&\quad + \int_0^y \mathcal{L}_{\mathcal{V}}(z)|\eta_z - \xi_z|_C\,dz\Big)\,dy + \int_0^x P\,dy \\
&\leq Q + \frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]|\eta(y) - \xi(y)|\,dy + \int_0^x \mathcal{L}_{\mathcal{U}}(y)\Big(|\eta_y - \xi_y|_C \\
&\quad + \int_0^y \mathcal{L}_{\mathcal{V}}(z)|\eta_z - \xi_z|_C\,dz\Big)\,dy + PX.
\end{aligned}$$

With the function γ in (12) we have

$$\gamma(x) \leq Q + PX + \frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]\gamma(y)\,dy + \int_0^x \mathcal{L}_{\mathcal{U}}(y)\left(\gamma(y) + \int_0^y \mathcal{L}_{\mathcal{V}}(z)\gamma(z)\,dz\right) dy$$

Applying the integral inequality from Theorem 3.2, with

$$u_0 = (Q + PX), \quad f(y) = \frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right],$$
$$g(x) = \mathcal{L}_{\mathcal{U}}(x) \quad \text{and} \quad h(x) = \mathcal{L}_{\mathcal{V}}(x)$$

we get

$$\gamma(x) \leq (Q + PX)\left[\exp\left(\int_0^x \frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-z)^\varrho\right] dz\right)\right.$$

$$+\int_0^x L_{\mathcal{U}}(y)\exp\left(\int_0^y\left[\frac{B(\varrho)}{1-\varrho}\mathbb{E}_\varrho\left(-\frac{\varrho}{1-\varrho}(x-z)^\varrho\right)+L_{\mathcal{U}}(z)+L_{\mathcal{V}}(z)\right]dz\right)$$
$$\times\exp\left(\int_y^x\frac{B(\varrho)}{1-\varrho}\mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]dz\right)dy\Bigg],$$

which gives the desired inequality (19). ■

To prove the results related to parameters, we consider the following equations involving the parameter of the form:

$$\frac{d}{dx}\eta(x)+{}^{ABR}{}_0\mathcal{D}_x^\varrho\eta(x)=\mathcal{U}\left(x,\rho_1,\eta_x,\int_0^x\mathcal{V}(x,y,\eta_y)\,dy\right),\quad x\in I=[0,X],\ X>0,\quad (20)$$

$$\frac{d}{dx}\eta(x)+{}^{ABR}{}_0\mathcal{D}_x^\varrho\eta(x)=\mathcal{U}\left(x,\rho_2,\eta_x,\int_0^x\mathcal{V}(x,y,\eta_y)\,dy\right),\quad x\in I=[0,X],\ X>0,\quad (21)$$

subject to

$$\eta(x)=\varsigma(x),\quad x\in[-r,0].\quad (22)$$

The following theorem shows the dependency of solution on parameters.

Theorem 5.4 (Dependence on parameters): *Let hypothesis* (A1)& (ii) *hold. Suppose that there exist* $\Omega,\ \Omega_1>0$ *and* $p\in C(J,\mathbb{R}_+)$ *such that*

$$|\mathcal{U}(x,\rho_1,\zeta_1,\eta_1)-\mathcal{U}(x,\rho_1,\zeta_2,\eta_2)|\le\Omega_1p(\tau)\left(|\zeta_1-\zeta_2|_C+|\eta_1-\eta_2|\right),$$
$$|\mathcal{U}(x,\rho_1,\zeta_1,\eta_1)-\mathcal{U}(x,\rho_2,\zeta_1,\eta_1)|\le\Omega|\rho_1-\rho_2|$$

where for all $x\in J,\rho_1,\rho_2\in\mathbb{R},\ \zeta_1,\zeta_2\in C$ *and* $\eta_1,\eta_2\in\mathcal{X}$. *If* η_1 *is the solution of problem* (20) *subject to* (22) *and* η_1 *is the solution of problem* (21) *subject to* (22) *then*

$$|\eta(x)-\xi(x)|$$
$$\le X\Omega\,|\rho_1-\rho_2|\left[\exp\left(\int_0^X\frac{B(\varrho)}{1-\varrho}\mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-z)^\varrho\right]dz\right)\right.$$
$$+\int_0^X\Omega_1p(y)\exp\left(\int_0^y\left[\frac{B(\varrho)}{1-\varrho}\mathbb{E}_\varrho\left(-\frac{\varrho}{1-\varrho}(x-z)^\varrho\right)+\Omega_1p(z)+L_{\mathcal{V}}(z)\right]dz\right)$$
$$\left.\times\exp\left(\int_y^X\frac{B(\varrho)}{1-\varrho}\mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]dz\right)dy\right].\quad (23)$$

Proof: We find for any $x\in I$

$$|\eta_1(x)-\eta_2(x)|$$
$$\le\frac{B(\varrho)}{1-\varrho}\int_0^x\mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]|\eta_1(y)-\eta_2(y)|\,dy$$
$$+\int_0^x\left|\mathcal{U}\left(y,\rho_1,(\eta_1)_y,\int_0^y\mathcal{V}(y,z,(\eta_1)_z)\,dz\right)\right.$$
$$\left.-\,\mathcal{U}\left(y,\rho_2,(\eta_2)_y,\int_0^y\mathcal{V}(y,z,(\eta_2)_z)\,dz\right)\right|dy$$

$$
\begin{aligned}
&\leq \frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]|\eta_1(y)-\eta_2(y)|\,dy \\
&\quad + \int_0^x \left|\mathcal{U}\left(y,\rho_1,(\eta_1)_y,\int_0^y \mathcal{V}(y,z,(\eta_1)_z)\,dz\right)\right. \\
&\quad \left. -\mathcal{U}\left(y,\rho_1,(\eta_2)_y,\int_0^y \mathcal{V}(y,z,(\eta_2)_z)\,dz\right)\right| dy \\
&\quad + \int_0^x \left|\mathcal{U}\left(y,\rho_1,(\eta_2)_y,\int_0^y \mathcal{V}(y,z,(\eta_2)_z)\,dz\right)\right. \\
&\quad \left. -\mathcal{U}\left(y,\rho_2,(\eta_2)_y,\int_0^y \mathcal{V}(y,z,(\eta_2)_z)\,dz\right)\right| dy \\
&\leq \frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]|\eta_1(y)-\eta_2(y)|\,dy + \int_0^x \Omega_1 p(y)\,\Big(|(\eta_1-\eta_2)(y)|_C \\
&\quad + \int_0^y L_\mathcal{V}(z)|(\eta_1-\eta_2)(z)|_C\,dz\Big)\,dy + \int_0^x \Omega\,|\rho_1-\rho_2|\,dy \\
&\leq \frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]|\eta_1(y)-\eta_2(y)|\,dy + \int_0^x \Omega_1 p(y)\,\Big(|(\eta_1-\eta_2)(y)|_C \\
&\quad + \int_0^y L_\mathcal{V}(z)|(\eta_1-\eta_2)(z)|_C\,dz\Big)\,dy + X\Omega\,|\rho_1-\rho_2|.
\end{aligned}
$$

With the function γ in (12) we have

$$
\begin{aligned}
\gamma(x) \leq X\Omega\,|\rho_1-\rho_2| &+ \frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]\gamma(y)\,dy \\
&+ \int_0^x \Omega_1 p(y)\left(\gamma(y)+\int_0^y L_\mathcal{V}(z)\gamma(z)\,dz\right)dy
\end{aligned}
$$

Applying the integral inequality from Theorem 3.2, with

$$
u_0 = X\Omega\,|\rho_1-\rho_2|,\quad f(y) = \frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right],
$$

$$
g(x) = \Omega_1 p(x) \quad \text{and} \quad h(x) = L_\mathcal{V}(x)
$$

we get

$$
\begin{aligned}
\gamma(x) \leq X\Omega\,|\rho_1-\rho_2| &\left[\exp\left(\int_0^x \frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-z)^\varrho\right]dz\right)\right. \\
&+ \int_0^x \Omega_1 p(y)\exp\left(\int_0^y \left[\frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left(-\frac{\varrho}{1-\varrho}(x-z)^\varrho\right)+\Omega_1 p(z)+L_\mathcal{V}(z)\right]dz\right) \\
&\left.\times \exp\left(\int_y^x \frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]dz\right)dy\right],
\end{aligned}
$$

which gives the desired inequality (19).

6. Ulam–Hyers–Rassias stability:

Definition 6.1: Equation (30) is said to be generalized Ulam–Hyers–Rassias stable with respect to the positive nondecreasing continuous function $\Phi : [-r, b] \to \mathbb{R}_+$ if there exists $C_\Phi > 0$ such that for each $\epsilon > 0$ and for each solution $\xi \in \mathcal{B}$ of following inequality

$$\left|\frac{d}{dx}\xi(x) + {}^{ABR}{}_0\mathcal{D}_x^{\varrho}\xi(x) - \mathcal{U}\left(x, \xi_x, \int_0^x \mathcal{V}(x, y, \xi_y)\,dy\right)\right| \le \epsilon\,\Phi(x), \quad x \in I \tag{24}$$

there is a solution $\eta \in \mathcal{B}$ of (30) with $|\xi(x) - \eta(x)| \le \epsilon\, C_\Phi\, \Phi(x)$ for $x \in [-r, b]$.

Theorem 6.1: *If the functions $\mathcal{U}$ and $\mathcal{V}$ satisfy the assumption (A1), then Equation (30) is Ulam–Hyers–Rassias that is stable with respect to Φ, provided $\int_0^x \Phi(t)\,dt \le \lambda\Phi(x)$, $x \in I$ and $\lambda > 0$.*

Proof: Denote that $\eta \in \mathcal{B}$ is the unique solution of the following problem

$$\frac{d}{dx}\eta(x) + {}^{ABR}{}_0\mathcal{D}_x^{\varrho}\eta(x) = \mathcal{U}\left(x, \eta_x, \int_0^x \mathcal{V}(x, y, \eta_y)\,dy\right), \; x \in I = [0, X],\ X > 0, \tag{25}$$

$$\eta(x) = \xi(x) \in [-r, 0], \tag{26}$$

Let $\xi \in \mathcal{B}$ be the solution of Equation (24). Then there exist $k_\xi(x) \in \mathcal{B}$ (which depend on ξ) such that $|k_\xi(x)| \le \epsilon\Phi(x)$ and

$$\frac{d}{dx}\xi(x) + {}^{ABR}{}_0\mathcal{D}_x^{\varrho}\xi(x) = \mathcal{U}\left(x, \xi_x, \int_0^x \mathcal{V}(x, y, \xi_y)\,dy\right) + k_\xi(x), \quad x \in I.$$

From the above equation we can write

$$\begin{aligned}&\left|\eta(x) - \xi(0) + \frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]\eta(y)\,dy\right.\\ &\left.\quad - \int_0^x \mathcal{U}\left(y, \eta_y, \int_0^y \mathcal{V}(y, z, \eta_z)\,dz\right)dy\right|\\ &\le \int_0^x |k_\xi(y)|\,dy \le \int_0^x \epsilon\Phi(y)\,dy \le \epsilon\,\lambda\Phi(x), \quad x \in I.\end{aligned} \tag{27}$$

Observe that $|\eta(x) - \xi(x)| = 0, x \in [-r, 0]$. By using assumption (A1) and (27) for $x \in I$ we have

$$\begin{aligned}&|\xi(x) - \eta(x)|\\ &= \left|\xi(x) - \xi(0) + \frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]\eta(y)\,dy\right.\\ &\left.\quad - \int_0^x \mathcal{U}\left(y, \eta_y, \int_0^y \mathcal{V}(y, z, \eta_z)\,dz\right)dy\right|\\ &\le \left|\xi(x) - \xi(0) + \frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]\xi(y)\,dy\right.\\ &\left.\quad - \int_0^x \mathcal{U}\left(y, \xi_y, \int_0^y \mathcal{V}(y, z, \xi_z)\,dz\right)dy\right|\end{aligned}$$

$$+\frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]|\eta(y)-\xi(y)|\,dy+\int_0^x L_{\mathcal{U}}(y)\left(|\eta_y-\xi_y|_{\mathcal{C}}\right.$$
$$\left.+\int_0^y L_{\mathcal{V}}(z)|\eta_z-\xi_z|_{\mathcal{C}}\,dz\right)dy$$
$$\leq \epsilon\lambda\Phi(x)+\frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]|\eta(y)-\xi(y)|\,dy$$
$$+\int_0^x L_{\mathcal{U}}(y)\left(|\eta_y-\xi_y|_{\mathcal{C}}+\int_0^y L_{\mathcal{V}}(z)|\eta_z-\xi_z|_{\mathcal{C}}\,dz\right)dy.$$

With the function γ in (12) we have

$$\gamma(x)\leq \epsilon\lambda\Phi(x)+\frac{B(\varrho)}{1-\varrho}\int_0^x \mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]\gamma(y)\,dy$$
$$+\int_0^x L_{\mathcal{U}}(y)\left(\gamma(y)+\int_0^y L_{\mathcal{V}}(z)\gamma(z)\,dz\right)dy.$$

Applying the integral inequality from Theorem 3.2, with

$$n(x)=\epsilon\Phi(x),\ f(y)=\frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right],\quad g(x)=L_{\mathcal{U}}(x)\quad\text{and}\quad h(x)=L_{\mathcal{V}}(x),$$

we get

$$|\eta(x)-\xi(x)|\leq \epsilon\lambda\Phi(x)\left[\exp\left(\int_0^x \frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-z)^\varrho\right]dz\right)\right.$$
$$+\int_0^x L_{\mathcal{U}}(y)\exp\left(\int_0^y\left[\frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left(-\frac{\varrho}{1-\varrho}(x-z)^\varrho\right)\right.\right.$$
$$\left.\left.+L_{\mathcal{U}}(z)+L_{\mathcal{V}}(z)\right]dz\right)$$
$$\left.\times\exp\left(\int_y^x \frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]dz\right)dy\right]. \tag{28}$$

Taking

$$C_\Phi=\lambda\left[\exp\left(\int_0^x \frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-z)^\varrho\right]dz\right)\right.$$
$$+\int_0^x L_{\mathcal{U}}(y)\exp\left(\int_0^y\left[\frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left(-\frac{\varrho}{1-\varrho}(x-z)^\varrho\right)+L_{\mathcal{U}}(z)+L_{\mathcal{V}}(z)\right]dz\right)$$
$$\left.\times\exp\left(\int_y^x \frac{B(\varrho)}{1-\varrho}\,\mathbb{E}_\varrho\left[-\frac{\varrho}{1-\varrho}(x-y)^\varrho\right]dz\right)dy\right]. \tag{29}$$

Then we have the following from (28)

$$|\eta(x)-\xi(x)|\leq \epsilon\, C_\Phi\,\Phi(x),\quad x\in I,$$

which shows that (30) is Ulam–Hyers–Rassias stable with respect to Φ.

7. Examples

Example 7.1: Considering a nonlinear multi-derivative fractional Volterra integro-differential equation:

$$\frac{d}{dx}\eta(x) + {}^{ABR}{}_{0}\mathcal{D}_{x}^{1/2}\eta(x) = \mathcal{U}\left(x, \eta_{x}, \int_{0}^{x} \mathcal{V}(x, y, \eta_{y})\, dy\right), \quad x \in J = [0, X],\ X > 0, \tag{30}$$

$$\eta(x) = \varsigma(x), \quad x \in [-r, 0], \tag{31}$$

Let

$$V(x, t, \eta_t) = K(x, t)\frac{sin|\eta'(x)|}{50 + x},$$

$$\varsigma(x) = \frac{sin(x)}{30 + x^2}.$$

Here, $X = 1, r = -1$.

Theorem 4.2 satisfies the following L_1, $L_2 = \frac{1}{50}$, $M_u = 0$, $M_v = 0$. after calculation. Thus a given system in light of Theorem 6.1 is UHR stable.

8. Conclusion

Volterra integro-differential equations, which combine differential and integral operators, are useful for simulating real-world problems. There are several research studies on volterra integro-differential equations in the literature, including some new work on fractional integro-differential equations. Many theories regarding such equations have been developed, and they have also been numerically solved using various approaches. As a result, we'll need to develop new equations to do a fresh research. As a result, in this paper, we proposed a novel class of volterra integro-differential equations with mixed operators. We deal with a nonlinear multi-derivative fractional volterra integro-differential equation with finite delay involving nonsingular Mittag–Leffler kernel. The existence results are obtained by using Krasnoselskii's fixed point theorem. Also we establish an inequality which is the generalization of Pachapate's one of the inequality, through which we have obtained the uniqueness results, Ulam-type stability and data dependence results.

Acknowledgments

The authors Aziz Khan and Thabet Abdeljawad would like to thank Prince Sultan university for giving the APC and the support through TAS research lab. A.K. investigated the manuscript. P.S. wrote the main manuscript. S.T. made a formal analysis of the manuscript. T.A dealt with the conceptualization. All authors read and approved the final manuscript.

Disclosure statement

No potential conflict of interest was reported by the author(s).

References

[1] Miller K, Ross B. An introduction to the fractional calculus and fractional differential equations. New York (NY): John Wiley & Sons; 1993.

[2] Podlubny I. Fractional differential equations. San Diego (CA): Academic Press; 1999.
[3] Caputo M, Fabrizio M. A new definition of fractional derivative without singular kernel. Prog Fract Differ Appl. 2015;1:73–85. DOI: 10.12785/pfda/010201
[4] Piotr H. Sobolev spaces on an arbitrary metric space. Potential Anal. 1996;5:403–415. DOI: 10.1007/BF00275475
[5] Atangana A, Baleanu D. Caputo-Fabrizio derivative applied to groundwater flow within confined aquifer. J Eng Mech. 2017;143(5):D4016005. DOI:10.1061/(ASCE)EM.1943-7889.0001091
[6] Jagdev S. Analysis of fractional blood alcohol model with composite fractional derivative. Chaos Solitons Fractals. 2020;140:110127. DOI:10.1016/j.chaos.2020.110127
[7] Yadav MP, Agarwal R, Purohit SD, et al. Groundwater flow in karstic aquifer: analytic solution of dual-porosity fractional model to simulate groundwater flow. Appl Math Sci Eng. 2022;30(1):598–608. DOI:10.1080/27690911.2022.2117913
[8] Singh J, Ahmed MA, Shaher M, et al. Computational analysis of fractional diffusion equations occurring in oil pollution. Mathematics. 2022;10(20):38–27. DOI:10.3390/math10203827
[9] Aydogan M, Baleanu D, Mousalou A, et al. On high order fractional integro-differential equations including the Caputo-Fabrizio derivative. Bound Value Probl. 2018;2018(1):1–5. DOI:10.1186/s13661-018-1008-990.
[10] Baleanu D, Jajarmi A, Mohammadi H, et al. A new study on the mathematical modelling of human liver with Caputo-Fabrizio fractional derivative. Chaos Solitons Fractals. 2020;134:109705. DOI:10.1016/j.chaos.2020.109705
[11] Abdeljawad T, Baleanu D. Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel. J Nonlinear Sci Appl. 2017;10(3):1098–1107. DOI:10.22436/jnsa
[12] Rezapour S, Etemad S, Mohammadi H. A mathematical analysis of a system of Caputo-Fabrizio fractional differential equations for the anthrax disease model in animals. Adv Differ Equ. 2020;2020(1):1–30. DOI:10.1186/s13662-019-2438-0
[13] Mohammadi H, Kumar S, Rezapour S, et al. A theoretical study of the Caputo-Fabrizio fractional modeling for hearing loss due to mumps virus with optimal control. Chaos Solitons Fractals. 2021;144:110668. DOI:10.1016/j.chaos.2021.110668
[14] Khan H, Alam K, Gulzar H, et al. A case study of fractal-fractional tuberculosis model in China: existence and stability theories along with numerical simulations. Math Comput Simul. 2022;198:455–473. DOI:10.1016/j.matcom.2022.03.009
[15] Etemad S, Avci I, Kumar P, et al. Some novel mathematical analysis on the fractal–fractional model of the AH1N1/09 virus and its generalized Caputo-Type version. Chaos Solitons Fractals. 2022;162:112511. DOI:10.1016/j.chaos.2022.112511
[16] Matar MM, Abbas MI, Alzabut J, et al. Investigation of the p-Laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives. Adv Differ Equ. 2021;1:1–18. DOI: 10.1186/s13662-021-03228-9
[17] Baleanu D, Jajarmi A, Hajipour M. A new formulation of the fractional optimal control problems involving Mittag–Leffler nonsingular kernel. J Optim Theory Appl. 2017;175:718–737. DOI:10.1007/s10957-017-1186-0
[18] Tuan NH, Mohammadi H, Rezapour S. A mathematical model for COVID-19 transmission by using the Caputo fractional derivative. Chaos Solitons Fractals. 2020;140:110107. DOI:10.1016/j.chaos.2020.110107
[19] Jarad F, Abdeljawad T, Hammouch Z. On a class of ordinary differential equations in the frame of Atangana-Baleanu fractional derivative. Chaos Solitons Fractals. 2018;117:16–20. DOI:10.1016/j.chaos.2018.10.006
[20] Kucche KD, Sutar ST. Analysis of nonlinear fractional differential equations involving Atangana-Baleanu-Caputo derivative. Chaos Solitons Fractals. 2021;143:110556. DOI:10.1016/j.chaos.2020.110556
[21] Ulam SM. A collection of the mathematical problems. New York: Interscience Publ.; 1960.
[22] Wang J, Feckan M, Zhou Y. Ulam's type stability of impulsive ordinary differential equations. J Math Anal Appl. 2012;395:258–264. DOI:10.1016/j.jmaa.2012.05.040

[23] Kucche KD, Sutar ST. Stability via successive approximation for nonlinear implicit fractional differential equations. MJPAA. 2017;3(1):36–55. DOI: 10.1515/mjpaa-2017-0004
[24] Kucche KD, Shikhare PU. Ulam-Hyers stability of integrodifferential equations in Banach spaces via Pachpatte's inequality. Asian Eur J Math. 2018;11(2):1850062. DOI:10.1142/S1793557118500626
[25] Kucche KD, Trujillo JJ. Theory of system of nonlinear fractional differential equations. Progr Fract Differ Appl. 2017;3:7–18. DOI:10.18576/pfda/030102
[26] Mohamed AS, Mahmoud RA. Picard, domain and predictor-corrector methods for an initial value problem of arbitrary (fractional) orders differential equation. J Egypt Math Soc. 2016;24:165–170. DOI:10.1016/j.joems.2015.01.001
[27] Sutar ST, Kucche KD. Existence and data dependence results for fractional differential equations involving Atangana-Baleanu derivative. Rend Circ Mat Palermo Ser 2. 2022;71(2):647–663. DOI:10.1007/s12215-021-00622-w
[28] Kilbas AA, Saigo M, Saxena K. Generalized Mittag-Leffler function and generalized fractional calculus operators. Integral Transforms Spec Funct. 2004;15:31–49. DOI:10.1080/10652460310001600717
[29] Erdelyi A, Magnus W, Oberhettinger F, et al. Higher transcendental functions. Vol. I, New York-Toronto-London: McGraw-Hill; 1953.
[30] Burton TA. A fixed-point theorem of krasnoselskii. Appl Math Lett. 1998;11:85–88. DOI:10.1016/S0893-9659(97)00138-9B. G. Pachpatte, On certain Volterra integro-differential equations, *FACTA UNIVERSITATIS, Ser. Math. Inform.*, 23 (2008), 1–12.
[31] Pachpatte B. Inequalities for differential and integral equations. New York (NY): Academic Press; 1998.