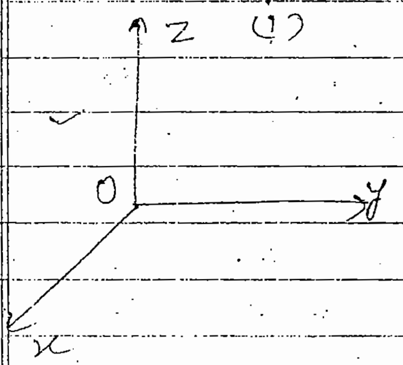
Books for EMT

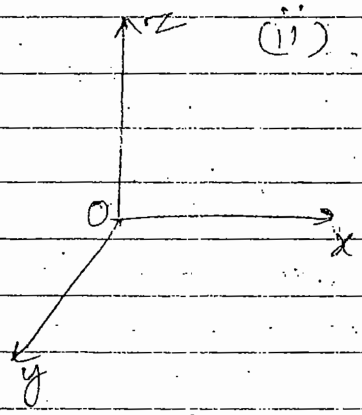
1) Griffiths - Introduction to electrodynamics
2) Sadiku - Electromagnetics
3) Chopra & Agarwal
4) J. D. Jackson

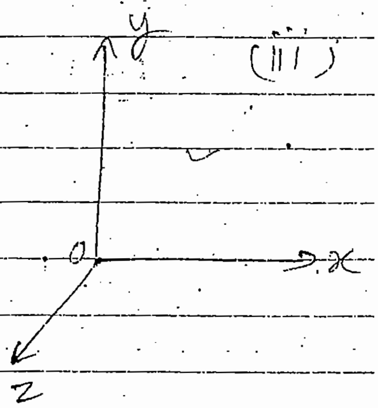
Mathematics for E.M.T.

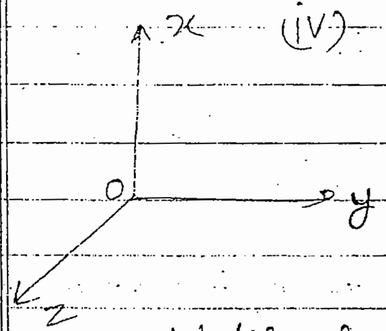
Vectors & Scalars

Scalar - which have only magnitude
Vector - dirn + magnitude









$(x, y, z) \rightarrow$ Orthogonal Co-ordinate system
Which are the notation correct in right handed system.
Ans $\rightarrow$ (i) & (iii) $\rightarrow$ correct

We'll always use R.H. system

$\hat{i} \times \hat{j} = \hat{k}$ $\quad\quad$ $\hat{i} \times \hat{k} = -\hat{j}$

$\hat{x} \times \hat{y} = \hat{z}$ $\quad\quad$ $\hat{k} \times \hat{j} = -\hat{i}$

$\hat{i} \longrightarrow \hat{j} \longrightarrow \hat{k} \longrightarrow \hat{i}$

In R.H.S. $\rightarrow$ Always move clockwise

Vectors

Polar Vector (True vector) (Vector) | Axial Vectors (pseudo vectors)

Polar Vectors are those vectors whose sence of direction is independent of handedness of the co-ordinate system.

or

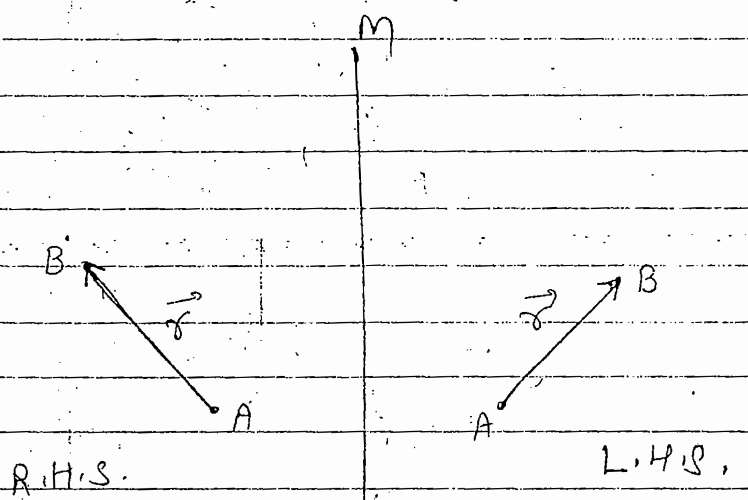
which change sign under inversion of co-ordinates (parity inversion)

Axial Vectors is opposite of polar vector i.e. whose sence of dirn depends on handedness of the co-ordinate system.

or

which does not change sign under inversion of co-ordinates

Suppose a vector defined from A to B in R.H.System. Put a mirror. Mirror image will be L.H.S.



Here ~~vector~~ sence of dirn is independent of handedness of co-ordinate system

Basic of polar vector → $\vec{r}$ → position vector.
Basic of pseudo (Axial) vector → $\vec{\omega}$ → angular velocity

How to Identify which one is polar or axial

$\vec{r}$ → Polar
$\vec{\omega}$ → Axial

Symmetry Operations
C, P, T

Parity Operations
On the basis of parity inversion we can define pseudo or true vector

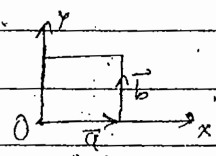
① Parity Inversion: $\vec{r} \to -\vec{r}$ $\qquad P: \vec{r} \to -\vec{r}$

$\vec{E} \xrightarrow{P} -\vec{E}$ (True vector)

If some vector changes sign under parity inv. then it is called True vector.
So electric field is a true vector

$\vec{B} \xrightarrow{P} +\vec{B}$ (not change sign)
Mag. field is pseudo vector

P (i) $\vec{r}$
P (ii) $\vec{v}$ $\quad \vec{v} = \omega \times r$
A (iii) $\vec{\omega}$
P (iv) $\vec{p}$ $\quad \vec{p} = m\vec{v}$
A (v) $\vec{L}$ (ang mom) $\quad \vec{L} = \vec{r} \times \vec{p}$
P (vi) $\vec{F}$ $\quad \vec{F} = m \times \vec{a} = m \times \frac{d^2\vec{r}}{dt^2}$
A (vii) $\vec{\tau}$ $\quad \vec{\tau} = \vec{r} \times \vec{F}$
A (viii) Area (vector Quantity) $\vec{a} \times \vec{b}$



P (ix) $\vec{J}$ $\quad \vec{J} = \rho\vec{v}$
P (x) $\vec{E}$ $\quad E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\vec{r}$
A (xi) $\vec{B}$ $\quad B = \frac{\mu_0}{4\pi}\int \frac{I d\vec{l} \times \vec{r}}{r^3}$

$\vec{B} = \vec{\nabla} \times \vec{A}$
A P (P)

P (xii) $\vec{A}$ (Mag. vector potⁿ)
P (xiii) $\vec{D}$ (Displacement vector)
P (xiv) $\vec{P}$ (Polarisation vector)
A (xv) $\vec{H}$ (Mag. field intensity)
A (xvi) $\vec{M}$ magnetization
P (xvii) $\vec{S}$ (pointing vector)

$+ \times + \rightarrow +$
$+ \times - \rightarrow -$
$- \times + \rightarrow -$
$- \times - \rightarrow +$

- sign → polar
+ " → pseudo (axial)

→ $\vec{\nabla}$ operator is polar vector

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\vec{\nabla} \xrightarrow{P} -\vec{\nabla}$$

→ Add or subtract - Polar to Polar
or Axial to Axial

$$\vec{D} = \epsilon_0 \underset{P}{\vec{E}} + \underset{P}{\vec{P}} \Rightarrow D \rightarrow \text{Polar}$$

$$\vec{B} = \mu_0 (\underset{A}{\vec{H}} + \underset{A}{\vec{M}})$$

$$\vec{S} = \underset{P}{\vec{E}} \times \underset{A}{\vec{H}}$$

For Polar Vectors parity operator value $p = -1$
Axial $p = +1$

So Parity operator has 2 eigen values.

(2) Time Reversal :-

entirely different symmetry operation.

(x, y, z, t)

$$T: \quad t \rightarrow -t$$

Position vector

$$\vec{r} \xrightarrow{T} \vec{r} \qquad \vec{\nabla} \xrightarrow{T} +\vec{\nabla}$$

$$\frac{\partial}{\partial t} \xrightarrow{T} \frac{-\partial}{\partial t}$$

If sign changes then $T = -1$
sign does not change $T = +1$

(i) $\vec{r} \xrightarrow{T} \vec{r}$

(ii) $\vec{v} \xrightarrow{T} -\vec{v}$ $\qquad \vec{v} = \frac{d\vec{r}}{dt}$

(iii) $\vec{a} \xrightarrow{T} +\vec{a}$ $\qquad \vec{a} = d^2\vec{r}/dt^2$

(iv) $\vec{E} \xrightarrow{T} +\vec{E}$

(v) $\vec{B} \xrightarrow{T} -\vec{B}$ $\qquad B = \frac{\mu_0}{4\pi}\int I \frac{\vec{dl} \times \vec{r}}{r^3}$, $\quad I = \frac{\partial q}{\partial t}$

(vi) $\rho \xrightarrow{T} +\rho$ $\qquad \rho = \frac{q}{V}$

(vii) $J \xrightarrow{T} -\vec{J}$ $\qquad \vec{J} = \rho\vec{v}$

(viii) $\vec{D} \xrightarrow{T} +\vec{D}$

(ix) $P \xrightarrow{T} +\vec{P}$

(x) $A \xrightarrow{T} -\vec{A}$ $\qquad B = \nabla \times A$

(xi) $\vec{H} \xrightarrow{T} -\vec{H}$

(xii) $\vec{M} \xrightarrow{T} -\vec{M}$

(xiii) $\vec{S} \xrightarrow{T} -\vec{S}$

(3) Charge Conjugation :-

$$C: \quad q \rightarrow -q$$

$$\vec{\nabla} \xrightarrow{C} +\vec{\nabla}$$

$$\frac{\partial}{\partial t} \xrightarrow{C} +\frac{\partial}{\partial t}$$

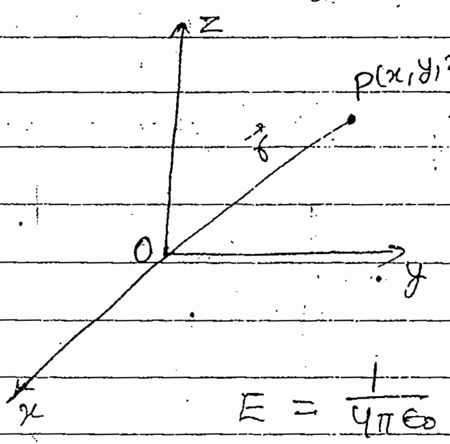
$r \xrightarrow{c} r$
$v \xrightarrow{c} v$
$a \xrightarrow{c} a$
$\rho \xrightarrow{c} -\rho$
$\vec{J} \xrightarrow{c} -\vec{J}$
$\vec{E} \xrightarrow{c} -\vec{E}$
$\vec{B} \xrightarrow{c} -\vec{B}$
$\vec{D} \xrightarrow{c} -\vec{D}$
$\vec{P} \xrightarrow{c} -\vec{P}$
$\vec{A} \xrightarrow{c} -\vec{A}$
$\vec{H} \xrightarrow{c} -\vec{H}$
$\vec{M} \xrightarrow{c} -\vec{M}$
$\vec{S} \xrightarrow{c} +\vec{S}$

* Maxwell's eqn's remains invariant under C, P, T operations.

Different types of Co-ordinate Systems:-

(i) Cartesian
(ii) spherical-polar
(iii) Cylendrical

(i) Cartesian Co-ordinate System:- $\hat{x}, \hat{y}, \hat{z}$
→ Three mutually perpendicular vectors
→ It is orthogonal co-ord system.



$$\vec{r} = (x-0)\hat{i} + (y-0)\hat{j} + (z-0)\hat{k}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

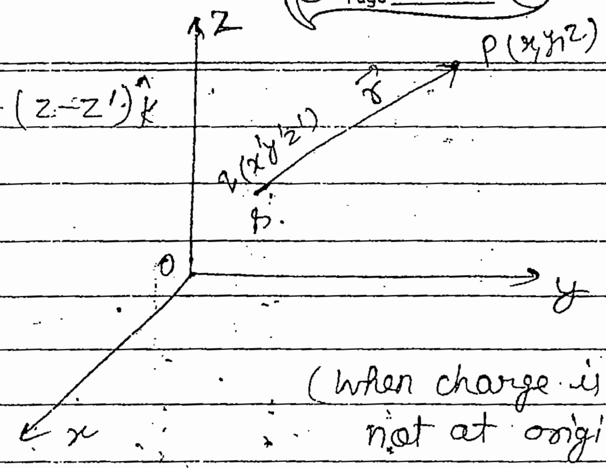
$$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\vec{r}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{(x^2+y^2+z^2)^{3/2}}(x\hat{i}+y\hat{j}+z\hat{k})$$

Limits → $x : -\infty \rightarrow +\infty$
$y : -\infty \rightarrow +\infty$
$z : -\infty \rightarrow +\infty$

$$\vec{r} = (x-x')\hat{i} + (y-y')\hat{j} + (z-z')\hat{k}$$



(When charge is not at origin

Length Elements :-

(i) $d\vec{l}_x = dx\,\hat{i}$
(ii) $d\vec{l}_y = dy\,\hat{j}$
(iii) $d\vec{l}_z = dz\,\hat{k}$

Surface Elements :-

(i) $d\vec{S}_x = d\vec{l}_y \times d\vec{l}_z = dy\,dz\,\hat{x}$

for flux calculation, "dot product will survive only if both vectors should have same dir^n or angle b/w them is not 90° $\int \underbrace{\vec{E}\cdot d\vec{S}}_{\text{same dir}^n}$

(ii) $d\vec{S}_y = d\vec{l}_z \times d\vec{l}_x = dx\,dz\,\hat{y}$
(iii) $d\vec{S}_z = d\vec{l}_x \times d\vec{l}_y = dx\,dy\,\hat{z}$

Volume Elements :-

$$V = \vec{a}\cdot(\vec{b}\times\vec{c})$$

Scalar which changes sign under inv^n of co-ordinate are Pseudo scalars.
Scalar which does not change sig under parity inv^n or co-ordinate inv^n are simply scalar.

Under parity inv^n a, b, c changes sign.
so volume → pseudo scalar

$$d\tau = d\vec{l}_x\cdot(d\vec{l}_y \times d\vec{l}_z)$$
$$d\tau = dx\,dy\,dz$$

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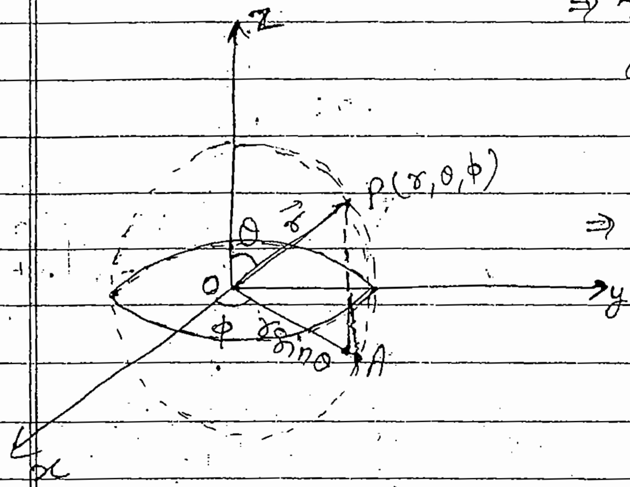
Spherical Polar co-ordinate system :-



⇒ $\vec{r}$ is radial vector & is always measured from the centre of sphere. It'll be same & all dir^n.

⇒ θ is angle b/w $\vec{r}$ & $\hat{z}$

ϕ is angle b/w $r\sin\theta$ & x axis.

$Z = r\cos\theta$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

Limits :-

r : 0 to ∞

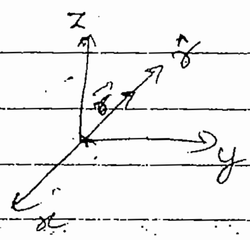
θ : $0 \to \pi$

ϕ : $0 \to 2\pi$

For Hemisphere , θ : $0 \to \pi/2$
ϕ : $0 \to 2\pi$

for 1/4 sphere , θ : $0 \to \pi/2$
ϕ : $0 \to \pi$

Length Elements :-



$\vec{dl_r} = dr\,\hat{r}$

$\hat{x} \to \hat{y} \to \hat{z} \to \hat{x}$

$\hat{x} \times \hat{y} = \hat{z}$

$\hat{x}\cdot\hat{x} = 1$, $\hat{x}\cdot\hat{y} = 0$

$\hat{r}\cdot\hat{r} = 1$

$\hat{r}\cdot\hat{\theta} = 0$

$\hat{r}\cdot\hat{\phi} = 0$

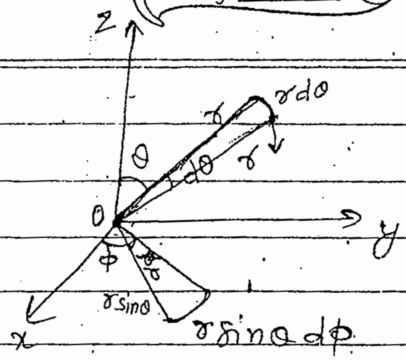
$\hat{r} \to \hat{\theta} \to \hat{\phi} \to \hat{r}$

$\hat{r} \times \hat{\theta} \Rightarrow \hat{\phi}$

dir^n of $\hat{r}$ is always away from the origin.

$+\hat{r} \to$ going away from origin

$-\hat{r} \to$ coming towards the origin



$\vec{dl_\theta} = r\,d\theta\,\hat{\theta}$ ~~(dirⁿ of change~~ clockwise)

~~$\vec{dl_\phi} = r\sin\theta\,d\phi\,\hat{\phi}$~~

(dirⁿ of change anticlockwise viewing from +ve z dirⁿ)

Surface Elements :- 3 surfaces.:. $r\theta$, $\theta\phi$, $r\phi$.

$\vec{dS_r} \rightarrow$ surface which does not change in r. r will be fix.

$\hat{r}$ always normal to surface

Outer spherical surface is $\vec{dS_r}$

$$\vec{dS_r} = r^2\sin\theta\,d\theta\,d\phi\,\hat{r} \qquad (\text{Sur. area} = 4\pi r^2)$$

$\vec{dS_\theta} \rightarrow$ surface where θ will not change.
2 dim. version of sphere is disc. for a disc, θ will not change.

$$\vec{dS_\theta} = r\sin\theta\,dr\,d\phi\,\hat{\theta}$$

$$= r\,dr\,d\phi\,\hat{\theta}$$

$\theta = 90°$ for disc
$r \rightarrow 0$ to r
$\phi \rightarrow 0$ to 2π
& get $dS_\theta = \pi r^2$

$$\int_0^r r\,dr\int_0^{2\pi} d\phi$$

$$\frac{r^2}{2} \times 2\pi = \pi r^2 \text{ (surface area)}$$

$$\vec{dS_\phi} = r\,dr\,d\theta\,\hat{\phi} \qquad (\text{least used})$$

Volume Element

$$d\tau = \vec{dl_r}\cdot(\vec{dl_\theta}\times\vec{dl_\phi})$$

$$d\tau = r^2\sin\theta\,d\theta\,d\phi\,dr$$

Que:- Calculate the volume of a sphere of radius R.

$$\tau = \int_0^R\int_0^\pi\int_0^{2\pi} d\tau$$

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$$\tau = \int_0^R \int_0^\pi \int_0^{2\pi} r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \left[\frac{r^3}{3}\right]_0^R \left[-\cos\theta\right]_0^\pi \left[\phi\right]_0^{2\pi}$$

$$= \frac{R^3}{3}\left[\cancel{-1}\,1+1\right]\left[2\pi\right]$$

$$= \frac{4}{3}\pi R^3$$

Relation b/w Unit Vectors :-

$\hat{r} \quad \hat{\theta} \quad \hat{\phi}$

$\downarrow$

$\hat{x} \quad \hat{y} \quad \hat{z}$

$z = r\cos\theta$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$$\boxed{r = \sqrt{x^2+y^2+z^2}} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi}$$

$$= r\sqrt{\cos^2\theta + \sin^2\theta(\cos^2\phi + \sin^2\phi)}$$

$$= r\sqrt{\cos^2\theta + \sin^2\theta}$$

$$= r$$

$$\theta = \cos^{-1}\frac{z}{r} \Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)}$$

$\phi = ?$

$$\frac{y}{x} = \tan\phi$$

$$\boxed{\phi = \tan^{-1}\left(\frac{y}{x}\right)}$$

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(6)

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\boxed{\vec{r} = r\sin\theta\cos\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\theta\,\hat{z}}$$

This is the funcⁿ of 3 variables r, θ, ϕ.

$\hat{r} \rightarrow$ dirⁿ of increment of r

$\hat{\theta} \rightarrow$ " " θ

$\hat{\phi} \rightarrow$ " " ϕ

$$d\vec{r} = \frac{\partial\vec{r}}{\partial r}dr + \frac{\partial\vec{r}}{\partial\theta}d\theta + \frac{\partial\vec{r}}{\partial\phi}d\phi$$

$\vec{r} = r\hat{r}$

$$\hat{r} = \frac{\frac{\partial\vec{r}}{\partial r}}{\left|\frac{\partial\vec{r}}{\partial r}\right|}$$

$$\frac{\partial\vec{r}}{\partial r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\left|\frac{\partial\vec{r}}{\partial r}\right| = \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}$$

$$= \sqrt{\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta}$$

$$= 1$$

$$\hat{r} = \frac{\partial\vec{r}}{\partial r} \Rightarrow \boxed{\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}} \quad (1)$$

$$\hat{\theta} = \frac{\frac{\partial\vec{r}}{\partial\theta}}{\left|\frac{\partial\vec{r}}{\partial\theta}\right|}$$

$$\frac{\partial\vec{r}}{\partial\theta} = r\cos\theta\cos\phi\,\hat{x} + r\cos\theta\sin\phi\,\hat{y} - r\sin\theta\,\hat{z}$$

$$\left|\frac{\partial\vec{r}}{\partial\theta}\right| = \sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta}$$

$$= r\sqrt{\cos^2\theta(\cos^2\phi + \sin^2\phi) + \sin^2\theta}$$

$$= r$$

$$\boxed{\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}} \quad (2)$$

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$$\hat{\phi} = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|}$$

$$\frac{\partial \vec{r}}{\partial \phi} = -r\sin\theta\sin\phi\,\hat{x} + r\sin\theta\cos\phi\,\hat{y}$$

$$\left|\frac{\partial \vec{r}}{\partial \phi}\right| = \sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi}$$

$$= r\sqrt{\sin^2\theta(1)}$$

$$= r\sin\theta$$

$$\boxed{\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}} \quad \text{——— (3)}$$

Using these ~~(3)~~ (1), (2), (3) eq^n convert spherical to cartisian.

eq^n (1) $\times \sin\theta$ } add
(2) $\times \cos\theta$ }

$$\sin\theta\,\hat{r} = \sin^2\theta\cos\phi\,\hat{x} + \sin^2\theta\sin\phi\,\hat{y} + \cos\theta\,\cancel{\sin\theta}\,\hat{z}$$

$$\cos\theta\,\hat{\theta} = \cos^2\theta\cos\phi\,\hat{x} + \cos^2\theta\sin\phi\,\hat{y} - \cos\cancel{\phi}\sin\theta\,\hat{z}$$

$$\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta} = \cos\phi\,\hat{x} + \sin^2\phi\,\hat{y} \quad \text{——— (4)}$$

eq^n (4) $\times \cos\phi$ − (3) $\times \sin\phi$,

$$\boxed{\sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi} = \hat{x}} \quad \text{——— (5)}$$

(4) $\times \sin\phi$ + (3) $\times \cos\phi$

$$\boxed{\sin\theta\sin\phi\,\hat{r} + \sin\phi\cos\theta\,\hat{\theta} + \cos\phi\,\hat{\phi} = \hat{y}} \quad \text{——— (6)}$$

(1) $\times \cos\theta$ − (2) $\times \sin\theta$

$$\cos\theta\sin\theta\cos\phi\,\hat{x} + \cos\theta\sin\theta\sin\phi\,\hat{y} + \cos^2\theta\,\hat{z} - \sin\theta\cos\theta\cos\phi\,\hat{x}$$
$$+ \sin\theta\cos\theta\sin\phi\,\hat{y} - \sin^2\theta\,\hat{z} = \hat{z}$$

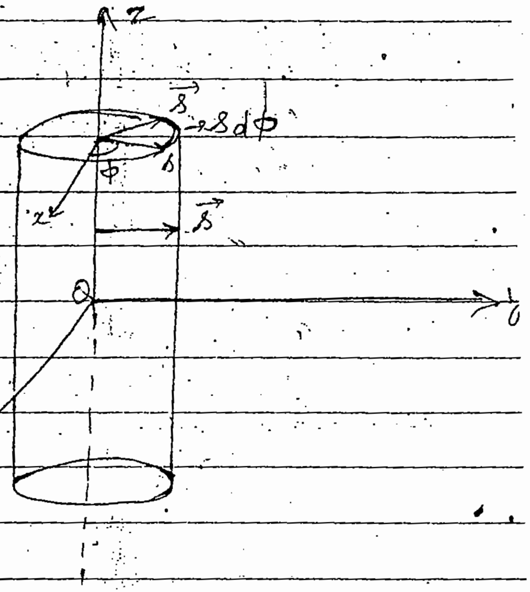
$$\boxed{\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}} \quad \text{——— (7)}$$

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$
$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$
$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\hat{x} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$$
$$\hat{y} = \sin\theta\sin\phi\,\hat{r} + \sin\phi\cos\theta\,\hat{\theta} + \cos\phi\,\hat{\phi}$$
$$\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

Cylendrical Co-ordinate system :-

s, ϕ, z are mutually orthogonal to eachother.



$\vec{s}$ is radius vector of the cylender.

ϕ → angle b/w $\vec{s}$ & x-axis.

$z = z$. It is length vector. There is no change in z.

$$x = s\cos\phi$$
$$y = s\sin\phi$$
$$Z = Z$$

radius vector ← $\vec{s} = s\,\hat{s}$ → unit vector

$\hat{s}$ will be +ve if move away from axis

" " −ve " " toward the axis.

$\hat{s}$ is the normal to the curved surface & +ve

Total Surface area $= \pi r^2 + 2\pi r l$

↓ cross sec. ↓ cylendrical curved

Limits :-

$s : 0 \rightarrow \infty$

$\phi : 0 \rightarrow 2\pi$

$z : -\infty \rightarrow +\infty$

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Length Elements :-

$$\vec{dl_s} = ds\,\hat{s}$$
$$\vec{dl_\phi} = s\,d\phi\,\hat{\phi}$$
$$\vec{dl_z} = dz\,\hat{z}$$

dir^n of change $\rightarrow \hat{s}, \hat{\phi}, \hat{z}$

Surface Element :-

dir^n of cylendrical curved surface is $\hat{s}$.

$$d\vec{S_s} = \vec{dl_\phi} \times \vec{dl_z}$$
$$= s\,d\phi\,dz\,(\hat{\phi} \times \hat{z})$$
$$\boxed{d\vec{S_s} = s\,d\phi\,dz\,\hat{s}}$$

$s \rightarrow \phi \rightarrow z \rightarrow s$

$\hat{x} \times \hat{y} = \hat{z}$

$\therefore \hat{s} * \hat{\phi} = \hat{z}$

$\hat{\phi} \times \hat{z} = \hat{s}$

$\hat{z} \times \hat{s} = \hat{\phi}$

When flux pass through this curved surface we use this element.

$s \rightarrow$ radius of cylender & remain const.

Dir^n of cross-sectional surface is $\hat{z}$ bcz it is normal to this surface.

$$d\vec{S_z} = \vec{dl_s} \times \vec{dl_\phi}$$
$$= s\,ds\,d\phi\,\hat{z}$$

If Press this cylender it'll become a rectangular sheet whose dir^n is $\hat{\phi}$ i.e. $\hat{\phi}$ is normal to this surface.

$$d\vec{S_\phi} = \vec{dl_z} \times \vec{dl_s}$$
$$= ds\,dz\,\hat{\phi}$$

Que: Suppose a cylender of radius s & height h. Calculate its surface area.

$$S_s = \int_S d\vec{S_s} = \int_S s\,d\phi\,dz$$

$$\int_S s\,ds\,d\phi \,,\quad \frac{s^2}{2} \cdot 2\pi$$

$$S_s = s\int_0^{2\pi} d\phi \int_{-h/2}^{h/2} dz = s[\phi]_0^{2\pi}[z]_{-h/2}^{h/2}$$

$$= s[2\pi]\left[\frac{h}{2}+\frac{h}{2}\right] = 2\pi s h$$

$$\boxed{S_s = 2\pi s h}$$

Q:- Calculate area of cross-sectional part.

$$S_z = \int_S s\, ds\, d\phi$$

$$= \int_0^s s\, ds \int_0^{2\pi} d\phi$$

$$= \left[\frac{s^2}{2}\right]_0^s [\phi]_0^{2\pi} = \frac{s^2}{2} 2\pi$$

$$S_z = \pi s^2$$

~~Volume~~

$$\cancel{S_\phi = \int_S ds\, dz}$$

$$\cancel{= \int_0^s ds \int}$$

Volume Element

$$d\tau = \vec{dl_s}\cdot(\vec{dl_\phi}\times\vec{dl_z})$$

$$\boxed{d\tau = s\, ds\, d\phi\, dz}$$

Relation b/w $\hat{s}, \hat{\phi}, \hat{z}$ & $\hat{x}, \hat{y}, \hat{z}$:-

$\hat{s} \rightarrow \hat{x}\ \hat{y}$

$\hat{\phi} \rightarrow \hat{x}\ \hat{y}$

$\hat{z} \rightarrow \hat{z}$

$$x = s\cos\phi \quad \text{---(1)}$$

$$y = s\sin\phi \quad \text{---(2)}$$

$$z = z \quad \text{---(3)}$$

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$eq^n (1)^2 + (2)^2 \Rightarrow \quad x^2 + y^2 = s^2(\cos^2\phi + \sin^2\phi)$

$$s^2 = x^2 + y^2$$

$$\boxed{s = \sqrt{x^2+y^2}} \quad \text{—— (4)}$$

Devide (2)/(1) $\Rightarrow \quad \tan\phi = \frac{y}{x}$

$$\boxed{\phi = \tan^{-1}\frac{y}{x}} \quad \text{—— (5)}$$

& $\boxed{z = z}$ —— (6)

$\cancel{s = \sqrt{x^2}}$ $\quad \vec{s} = x\hat{i} + y\hat{j}$

$$\boxed{\vec{s} = s\cos\phi\ \hat{i} + s\sin\phi\ \hat{j}} \quad \text{—— (7)}$$

$\vec{s}$ is depanding on s & ϕ.

$$\hat{s} = \frac{\frac{\partial\vec{s}}{\partial s}}{\left|\frac{\partial\vec{s}}{\partial s}\right|}, \quad \hat{\phi} = \frac{\frac{\partial\vec{s}}{\partial\phi}}{\left|\frac{\partial\vec{s}}{\partial\phi}\right|},$$

$$\frac{\partial\vec{s}}{\partial s} = \cos\phi\ \hat{i} + \sin\phi\ \hat{j}$$

$$\left|\frac{\partial\vec{s}}{\partial s}\right| = \sqrt{\cos^2\phi + \sin^2\phi} = 1$$

$$\boxed{\hat{s} = \cos\phi\ \hat{i} + \sin\phi\ \hat{j}} \quad \text{—— (8)}$$

$$\frac{\partial\vec{s}}{\partial\phi} = -s\sin\phi\ \hat{i} + s\cos\phi\ \hat{j}$$

$$\left|\frac{\partial\vec{s}}{\partial\phi}\right| = \sqrt{s^2\sin^2\phi + s^2\cos^2\phi} = s\sqrt{(1)}$$

$$= s$$

$$\boxed{\hat{\phi} = -\sin\phi\ \hat{i} + \cos\phi\ \hat{j}} \quad \text{—— (9)}$$

$$\boxed{\hat{z} = \hat{z}} \quad \text{—— (10)}$$

(9) × cos φ + (8) × sin φ ⇒

$$\sin\phi\cos\phi\,\hat{i} + \sin^2\phi\,\hat{j} = \hat{s}\sin\phi$$

$$-\sin\phi\cos\phi\,\hat{i} + \cos^2\phi\,\hat{j} = \hat{\phi}\cos\phi$$

add $$(\sin^2\phi + \cos^2\phi)\,\hat{j} = \hat{s}\sin\phi + \hat{\phi}\cos\phi$$

$$\boxed{\hat{j} = \hat{y} = \sin\phi\,\hat{s} + \cos\phi\,\hat{\phi}} \quad \text{——(11)}$$

eqⁿ (8) × cos φ − (9) × sin φ

$$\cos^2\phi\,\hat{i} + \sin\phi\cos\phi\,\hat{j} = \cos\phi\,\hat{s}$$

$$-\sin^2\phi\,\hat{i} \pm \cos\phi\sin\phi\,\hat{j} = \sin\phi\,\hat{\phi}$$

sub. $$\boxed{\hat{i} = \hat{x} = \cos\phi\,\hat{s} - \sin\phi\,\hat{\phi}} \quad \text{——(12)}$$

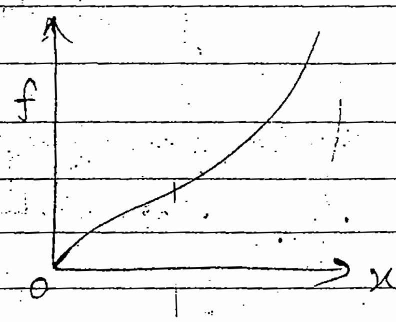
$$\boxed{\hat{z} = \hat{z}} \quad \text{——(13)}$$

Gradient, Divergance & Curl :-

Gradient :- It is defined for a scalar. We can not take grad of any vector.

suppose a fucⁿ varry with x, f(x)

then $$df = \underbrace{\left(\frac{\partial f}{\partial x}\right)}_{\text{slop}} dx$$

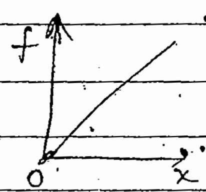


This derivative shows how f increases with x

This is the slop of the graph.

If graph linear then slop → const.

Non " → change



x → independent

f → dependent

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indi. variable x shows $\rightarrow$ change in f by changing x.

Consider a fun^n $T(x, y, z)$

Now T is changing with x, y, z. It can't describe by one variable

Its differential

$$dT = \left(\frac{\partial T}{\partial x}\right)dx + \left(\frac{\partial T}{\partial y}\right)dy + \left(\frac{\partial T}{\partial z}\right)dz$$

We know that

$$\vec{\nabla} \equiv \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

So we can write

$$\boxed{dT = \underbrace{(\vec{\nabla} T)}_{\text{gradient of fun}^n\text{ (temp.)}} \cdot d\vec{r}}$$

$\rightarrow$ gradient of a scalar is a vector quantity.

$$dT = \underbrace{|\vec{\nabla} T|}_{\text{magnitude of grad. of scalar}} dr \cos\theta$$

If $\theta = 0$,

then change is maximum

If $\theta = 90$, then change is minimum.

Dir^n of grad. of scalar shows dir^n of maximum increase of the fun^n. ~~The dir^n~~

e.g. $\vec{E} = -\vec{\nabla} V$

dir^n of electric field is in the dir^n of maximum decrease of potential (bcz there is $-$ sign.)

Let us consider a conducting sphere on which electric field is normal to the surface.

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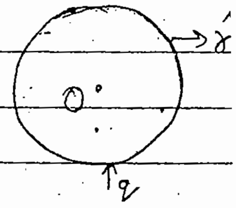
It is equi pot^n surface. So Elec. field is in the dir^n of max. dec. in pot^n.



$$\boxed{\bar{\nabla} T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}} \rightarrow \text{In Cartesian}$$

$$\boxed{\bar{\nabla} T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi}} \rightarrow \text{In spherical polar}$$

$$\boxed{\bar{\nabla} T = \left(\frac{\partial T}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\phi} + \frac{\partial T}{\partial z}\hat{z}\right)} \rightarrow \text{In cylindrical}$$

Now, general formula for gradient :-

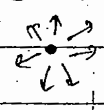
$$\boxed{\bar{\nabla}(r^n) = n r^{n-2}\vec{r} = n r^{n-1}\hat{r}} \qquad [\vec{r} = r\hat{r}]$$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

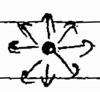
$\vec{r} = (x-x')\hat{i} + (y-y')\hat{j} + (z-z')\hat{k}$

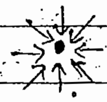
Divergence :- It is defined for a vector.
Divergence means spread (how much a vector can spread in space)
How much a vector diverges (spread) from a point in the space.
for source → +ve divergence
" sink → −ve "

Sun

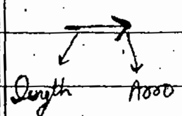


Outward dir is +ve   Inward is −ve



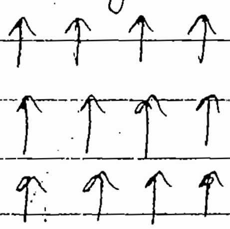


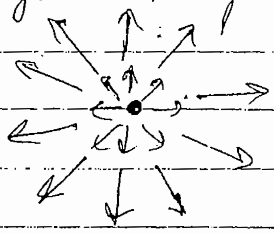
Graphical rep^n of Divergence



, Arrow shows dir^n & length shows magnitude.

If vector length is same & no change in magnitude then zero divergence.





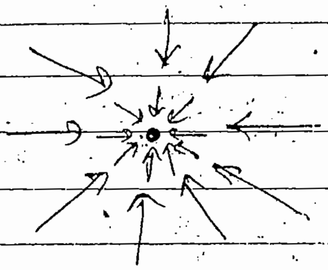
zero divergence

Non zero div. bcz length is changed.

$\downarrow$ +ve div

No Variation in mag. $\rightarrow$ Zero div

Variation in mag. $\rightarrow$ Non-zero div.



This is -ve divergence.

away from source $\rightarrow$ +ve

towards sink $\rightarrow$ -ve

If $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$

formula $\Rightarrow$ $\nabla \cdot \vec{A} = \left(\frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z \right)$

Variation of x comp. of A with x.

" y " y.

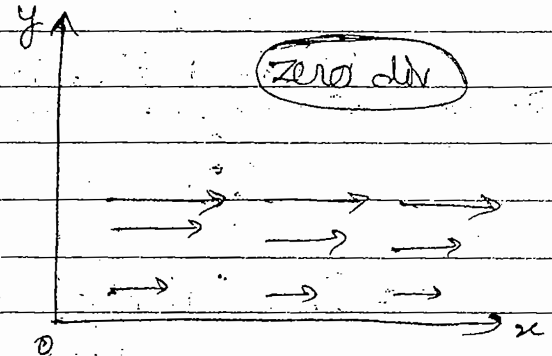
" z " z.

This is zero divergence.

dir^n of vector is x

& change in mag. is in y-dir^n.

There is no chang in length along x-dir^n.



zero div

Here $\nabla \cdot A = \frac{\partial}{\partial x} A_x$.

(Here Non-zero curl but zero div.)

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General formula

$$\boxed{\begin{aligned} \vec{\nabla}\cdot(r^n \hat{r}) &= (n+2)\, r^{n-1} \\ \vec{\nabla}\cdot(r^n \vec{r}) &= (n+3)\, r^{n} \end{aligned}}$$

This formula is applicable everywhere but except $\frac{\hat{r}}{r^2}$ i.e. $\frac{\vec{r}}{r^3}$. This is dirac delta funⁿ in Mathematics. It is always zero.

If div of a vector is zero then it is called $\boxed{\text{Solenoidal vector.}}$ $\boxed{\vec{\nabla}\cdot\vec{A}=0}$

Curl :- It is defined for a vector. Curl of a vector is also a vector. Curl is related to rotation i.e. How much a vector is rotates about a point in the space.
If there is no rotation then curl is zero.
If curl of a vector is zero then vector is irrotational.

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

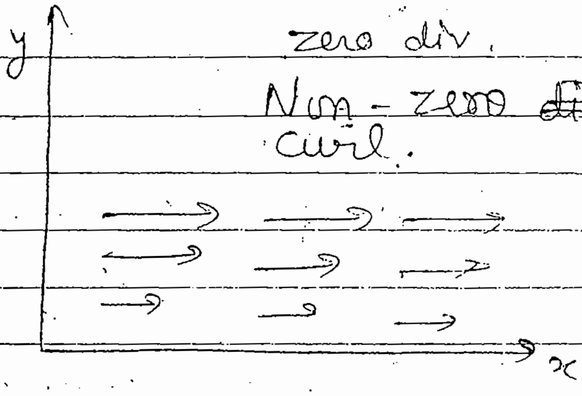
$$= \hat{i}\left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right) - \hat{j}\left(\frac{\partial}{\partial x}A_z - \frac{\partial}{\partial z}A_x\right) + \hat{k}\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right)$$

This is the curl in cartisian co-ordinate system.

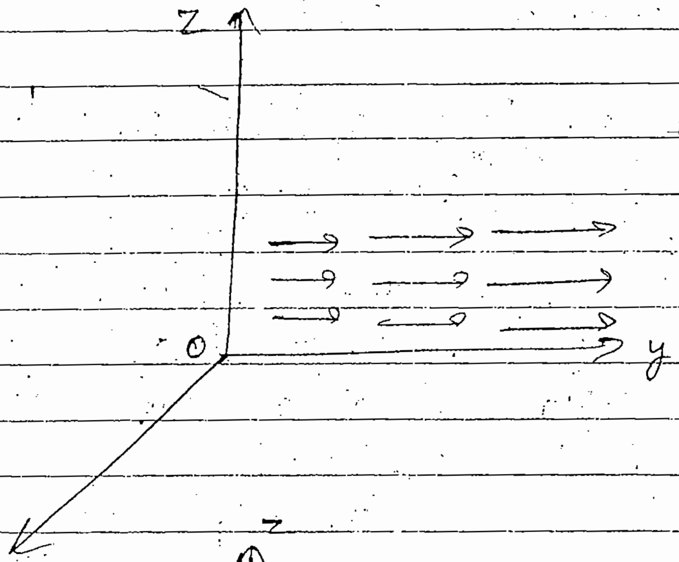
If vector in x-dirⁿ then it will change in perpendicular dirⁿ i.e. A_x change with y & z axis then curl Non-zero. but if A_x change with x-axis then curl zero.

Graphical Repⁿ :- vector changing mag. with y & pointing towards x- dirⁿ.

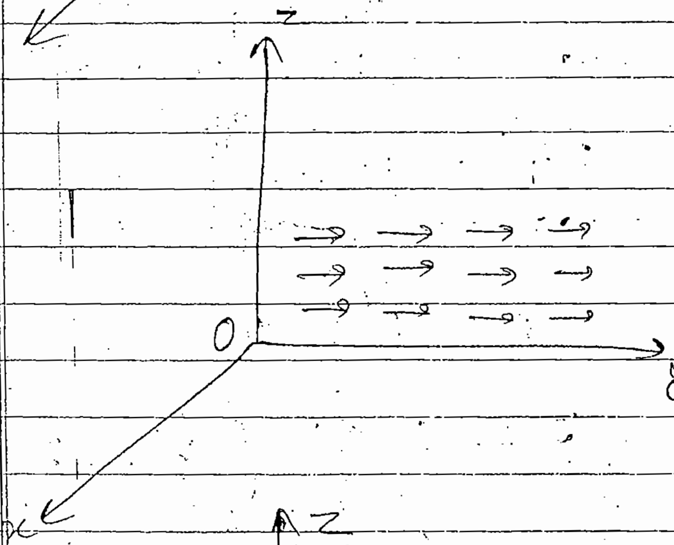
zero div.
Non - zero ~~div~~ curl.



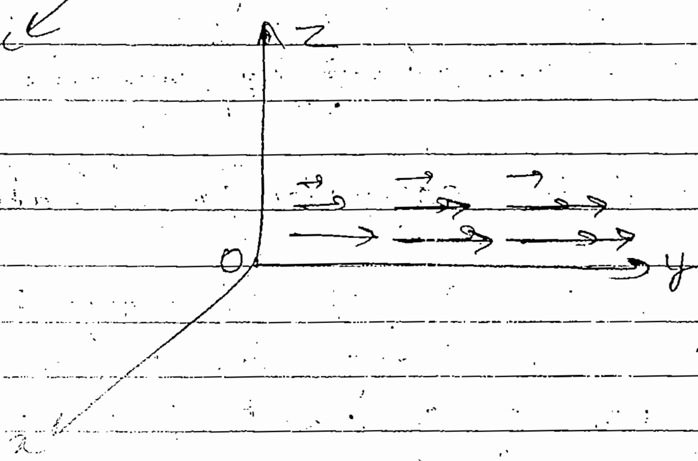
non zro div
zero curl



Zero div
Zero Curl



Non zero div
Non Zero curl



General formula :-

$$\boxed{\bar{\nabla} \times (r^n \hat{r}) = 0}$$

$$\boxed{\bar{\nabla} \times (r^n \vec{r}) = 0}$$

$$\boxed{\bar{\nabla} \times \vec{r} = 0}$$

formula's for divergence :-

Cartisian $\bar{\nabla} \cdot \bar{A} = \left(\frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z \right)$

$\vec{A} = \underset{\text{comp. of A along x}}{A_x \hat{i}} + \underset{\text{along y}}{A_y \hat{j}} + \underset{\text{along z}}{A_z \hat{k}}$, $\vec{A}$ in $\underset{\text{comp. of A along r}}{A_r}, A_\theta, A_\phi$

spherical Polar

$$\bar{\nabla} \cdot \bar{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta A_\theta) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} A_\phi$$

Cylendrical A_s, A_ϕ, A_z

$$\bar{\nabla} \cdot \bar{A} = \frac{1}{s} \frac{\partial}{\partial s} (s A_s) + \frac{1}{s} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

formula's for Curl :-

spherical Polar

$$\bar{\nabla} \times \bar{A} = \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial \theta} (\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial \phi} \right] \hat{r} + \frac{1}{r} \left[\frac{1}{\sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} (r A_\phi) \right] \hat{\theta} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \hat{\phi}$$

$$\frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r \sin\theta\, A_\phi \end{vmatrix}$$

Cylendrical

$$\bar{\nabla} \times \bar{A} = \left(\frac{1}{s} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{s} + \left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s} \right) \hat{\phi} + \frac{1}{s} \left[\frac{\partial}{\partial s} (s A_\phi) - \frac{\partial}{\partial \phi} A_s \right] \hat{z}$$

$$\frac{1}{s} \begin{vmatrix} \hat{s} & s\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_s & s A_\phi & A_z \end{vmatrix}$$

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Note:- Div, Grad, Curl all are first derivatives.

Laplacian :-

Laplacian is defined for both scalar & vector

$$\bar{\nabla}\cdot\bar{\nabla} = \nabla^2$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$A_x, A_y, A_z \rightarrow$ Scalars

Cartesian :-

$$\nabla^2 T = \left(\frac{\partial^2}{\partial x^2}T + \frac{\partial^2}{\partial y^2}T + \frac{\partial^2}{\partial z^2}T\right)$$

$T \rightarrow$ Scalar

for a vector,

$$\nabla^2\vec{A} = \left(\frac{\partial^2}{\partial x^2}A_x + \frac{\partial^2}{\partial y^2}A_x + \frac{\partial^2}{\partial z^2}A_x\right)\hat{i} + \left(\frac{\partial^2}{\partial x^2}A_y + \frac{\partial^2}{\partial y^2}A_y + \frac{\partial^2}{\partial z^2}A_z\right)\hat{j}$$
$$+ \left(\frac{\partial^2}{\partial x^2}A_z + \frac{\partial^2}{\partial y^2}A_z + \frac{\partial^2}{\partial z^2}A_z\right)\hat{k}$$

$\boxed{\nabla^2 V = 0}$ Laplacian eq^n

$\boxed{\nabla^2 V = -\rho/\epsilon_0}$ Poision eq^n

Spherical Polar

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}T\right)$$

Cylindrical

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2}$$

Table for General Formula

System	u	v	w	f	g	h
C	x	y	z	1	1	1
S.P.	r	θ	ϕ	1	r	$r\sin\theta$
Cy.	s	ϕ	z	1	s	1

(i) **Gradient :-**

$$\bar{\nabla} T = \frac{1}{f}\frac{\partial T}{\partial u}\hat{u} + \frac{1}{g}\frac{\partial T}{\partial v}\hat{v} + \frac{1}{h}\frac{\partial T}{\partial w}\hat{w}$$

(ii) **Divergence :-**

$$\bar{\nabla}\cdot\bar{A} = \frac{1}{fgh}\left[\frac{\partial}{\partial u}(gh A_u) + \frac{\partial}{\partial v}(fh A_v) + \frac{\partial}{\partial w}(fg A_w)\right]$$

(iii) **Curl :-**

$$\bar{\nabla}\times\bar{A} = \frac{1}{fgh}\begin{vmatrix} f\hat{u} & g\hat{v} & h\hat{w} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ fA_u & gA_v & hA_w \end{vmatrix}$$

(iv) **Laplacian :-**

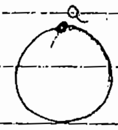
$$\nabla^2 T = \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\frac{\partial T}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{fh}{g}\frac{\partial T}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{fg}{h}\frac{\partial T}{\partial w}\right)\right]$$

Theoremes

1) <u>Gradient Theorem</u>

$$\int_a^b (\bar{\nabla} T) \cdot d\bar{r} = \int_a^b dT = T(b) - T(a)$$

$$\boxed{\oint (\bar{\nabla} T) \cdot d\bar{r} = 0}$$



bcz in close line integral initial & final points are same.

2) <u>Divergence Theorem / Gauss Theorem</u> is used to convert closed surface integral into volume integral.

$$\oint_S \bar{A} \cdot d\bar{s} = \int_V (\bar{\nabla} \cdot \bar{A}) \, d\tau$$

* Close surfaces are those surfaces which bounds some volume. It can be used for sphere, cylender, cube but not for a disc bcz it is not a closed surface.

3) <u>Curl Theorem / Stoke's Theorem</u> is used to convert closed line integral into open surface integral.

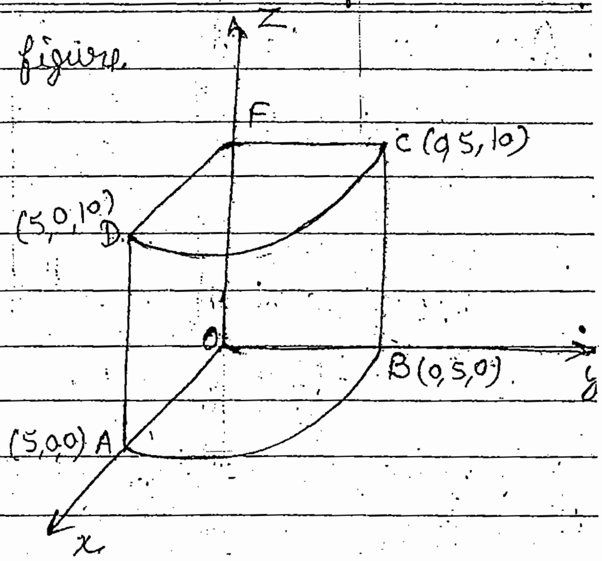
$$\oint_C \vec{A} \cdot d\vec{l} = \int_S (\bar{\nabla} \times \vec{A}) \cdot ds$$

* closed line curve are those which bound some surface.

πr^2 ✓ ——— × Not a closed line curve.

Que :- Cosider the object in the figure. Calculate ~~fix~~

(i) the length BC
(ii) the length CD
(iii) Surface area ABCD
(iv) Surface area ABO
(v) Surface area AOFD
(vi) Volume ABDCFO



(i) $BC = \int_0^{10} dz$

$= 10$

(ii) $CD = \int \rho\, d\phi$ $\qquad [\rho = 5]$

$= 5\int_0^{\pi/2} d\phi$

$= \frac{5\pi}{2}$

(iii) $ABCD = \int_S \rho\, d\phi\, dz$

$= 5\int_0^{\pi/2} d\phi \int_0^{10} dz = 5\left[\frac{\pi}{2}\right][10]$

$= 25\pi$

z-dim
cross-sec.

(iv) $ABO = \int_S \rho\, d\rho\, d\phi$

$= \int_0^5 \rho\, d\rho \int_0^{\pi/2} d\phi = \left[\frac{\rho^2}{2}\right]_0^5 \left[\phi\right]_0^{\pi/2}$

$= \frac{25}{2}\cdot\frac{\pi}{2} = \frac{25\pi}{4}$

area vector $\hat{\phi}$

(v) AOFD (area) $= \int_S d\rho\, dz$

$= \int_0^5 d\rho \int_0^{10} dz = (5)(10) = 50$

(vi) ABDCFO (Volume) $= \int_V \rho\, d\rho\, d\phi\, dz = \int_0^5 \rho\, d\rho \int_0^{\pi/2} d\phi \int_0^{10} dz$

$= \frac{25}{2}\cdot\frac{\pi}{2}\cdot 10 = \frac{125\pi}{2}$

Q.1:- (a) $f(x,y,z) = x^2 + y^3 + z^4$

$$\cancel{\text{grad } f} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= 2x\,\hat{i} + 3y^2\,\hat{j} + 4z^3\,\hat{k}$$

(b) $f(x,y,z) = x^2 y^3 z^4$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= 2xy^3z^4\,\hat{i} + 3x^2y^2z^4\,\hat{j} + 4x^2y^3z^3\,\hat{k}$$

$$= xyz\left(2y^2z^3\,\hat{i} + 3xyz^3\,\hat{j} + 4xy^2z^2\,\hat{k}\right)$$

(c) $f(x,y,z) = e^x \sin y \ln(z)$

$$\nabla f = e^x \sin y \ln(z)\,\hat{i} + e^x \cos y \ln(z)\,\hat{j} + \frac{e^x \sin y}{z}\,\hat{k}$$

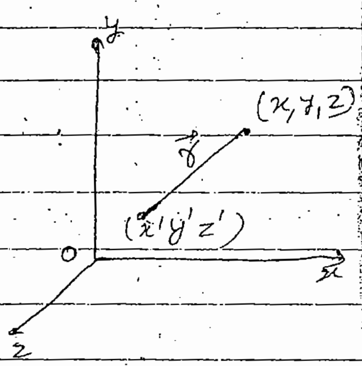
Q.2 (a) $\bar{\nabla}(r^2) = 2\vec{r}$

$$\bar{\nabla}(r^2) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2)$$

$$= 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$$

$$= 2(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= 2\vec{r}$$



(b) $\bar{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$

$$\nabla\left(\frac{1}{r}\right) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2)^{-1/2}$$

$$= -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}\,\cancel{2}x\,\hat{i} - \frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}\,2y\,\hat{j}$$

$$- \frac{1}{2}(x^2+y^2+z^2)^{-3/2}\,2z\,\hat{k}$$

$$= -(x^2+y^2+z^2)^{-3/2}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= -\frac{(x\hat{i} + y\hat{j} + z\hat{k})}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = -\frac{\vec{r}}{r^3} = \frac{-r\hat{r}}{r^3} = \frac{-\hat{r}}{r^2}$$

(c) $\nabla(r^n) = n r^{n-2} \vec{r}$

$$\nabla(r^n) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2)^{n/2}$$

$$= \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2x)\hat{i} + \cdots$$

$$= n(x^2+y^2+z^2)^{\frac{n}{2}-1}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= n\left[(x^2+y^2+z^2)^{1/2}\right]^{n-2}\vec{r}$$

$$= n r^{n-2}\vec{r}$$

Q.3 : (a) $\vec{v}_a = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}$

$$\text{div } v_a = \vec{\nabla}\cdot\vec{v}_a = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}\right)$$

$$= 2x - 2x = 0$$

(b) $\vec{v}_b = xy\hat{x} + 2yz\hat{y} + 3zx\hat{z}$

$$\vec{\nabla}\cdot\vec{v}_b = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(xy\hat{x} + 2yz\hat{y} + 3zx\hat{z}\right)$$

$$= y + 2z + 3x$$

(c) $\vec{v}_c = y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z}$

$$\vec{\nabla}\cdot\vec{v}_c = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(y^2\hat{x} + (2xy+z^2)\hat{y} + 2yz\hat{z}\right)$$

$$= 2x + 2y = 2(x+y)$$

Q.5 : (a) $T_a = x^2 + 2xy + 3z + 4$

$$\nabla^2 T_a = \left[\frac{\partial^2}{\partial x^2}(x^2+2xy+3z+4) + \frac{\partial^2}{\partial y^2}(x^2+2xy+3z+4) + \frac{\partial^2}{\partial z^2}(x^2+2xy+3z+4)\right]$$

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$$= \frac{\partial}{\partial x}(2x+2y) + \frac{\partial}{\partial y}(2x) + \frac{\partial}{\partial z}(3)$$

$$= 2$$

(b) $T_b = \sin x \sin y \sin z$

$$\nabla^2 T_b = \frac{\partial^2}{\partial x^2}(\sin x \sin y \sin z) + \frac{\partial^2}{\partial y^2}(\sin x \sin y \sin z) + \frac{\partial^2}{\partial z^2}(\sin x \sin y \sin z)$$

$$= \frac{\partial}{\partial x}(\cos x \sin y \sin z) + \frac{\partial}{\partial y}(\sin x \cos y \sin z) + \frac{\partial}{\partial z}(\sin x \sin y \cos z)$$

$$= -\sin x \sin y \sin z - \sin x \sin y \sin z - \sin x \sin y \sin z$$

$$= \sin x \sin y \sin z(-3)$$

$$= -3 \sin x \sin y \sin z$$

(c) $T_c = e^{-5x} \sin 4y \cos 3z$

$$\nabla^2 T_c = \frac{\partial^2}{\partial x^2}[e^{-5x} \sin 4y \cos 3z] + \frac{\partial^2}{\partial y^2}(e^{-5x} \sin 4y \cos 3z) + \frac{\partial^2}{\partial z^2}(e^{-5x} \sin 4y \cos 3z)$$

$$= 25 e^{-5x} \sin 4y \cos 3z - 16 e^{-5x} \sin 4y \cos 3z - 9 e^{-5x} \sin 4y \cancel{\sin 3z} \cos 3z$$

$$= e^{-5x} \sin 4y \cos 3z [25 - 16 - 9]$$

$$= 0$$

(d) $\vec{v} = x^2 \hat{x} + 3xz^2 \hat{y} - 2xz \hat{z}$

$$\nabla^2 \vec{v} = \left(\frac{\partial^2}{\partial x^2} x^2 + \frac{\partial^2}{\partial y^2} x^2 + \frac{\partial^2}{\partial z^2} x^2\right)\hat{x} + \left(\frac{\partial^2}{\partial x^2} 3xz^2 + \frac{\partial^2}{\partial y^2} 3xz^2 + \frac{\partial^2}{\partial z^2} 3xz^2\right)\hat{y}$$

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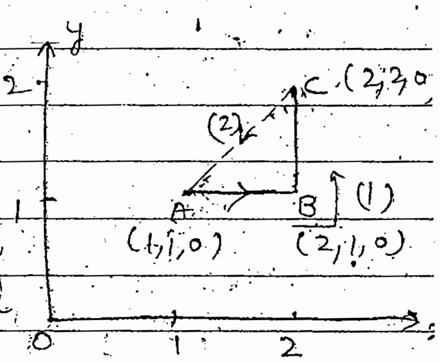
$$+\left(\frac{\partial^2}{\partial x^2}(-2xz)+\frac{\partial^2}{\partial y^2}(-2xz)+\frac{\partial^2}{\partial z^2}(-2xz)\right)\hat{z}$$

$$\nabla^2\vec{v} = 2\hat{x}+6x\hat{y}+0 = 2\hat{x}+6x\hat{y}$$

Q.6: $\vec{v} = y^2\hat{x}+2x(y+1)\hat{y}$

$a=(1,1,0)$, $b=(2,2,0)$

$$\oint \bar{v}\cdot d\bar{l} = \int_{AB}\bar{v}\cdot d\bar{l} + \int_{BC}\bar{v}\cdot d\bar{l} + \int_{CA}\bar{v}\cdot d\bar{l}$$



$$= \int_1^2 \cancel{\bar{v}\cdot d\bar{l}}\,(y^2\hat{x}+2x(y+1)\hat{y})\cdot dx\,\hat{x}$$

$$+\int_1^2 (y^2\hat{x}+2x(y+1)\hat{y})\cdot dy\,\hat{y}$$

$$+\int_2^1 (y^2\hat{x}+(2x(y+1)\hat{y})\cdot d\cancel{z}\,\hat{y}$$

$$= \int_1^2 y^2dx + \int_1^2 2x(y+1)dy + \int_2^1 \cancel{0\,dz} \quad (2x(y+1))dy$$

$$= y^2[x]_1^2 + 2x\left[\frac{y^2}{2}+y\right]_1^2 = y^2(2-1)+2x\left[\frac{4}{2}+2-\frac{1}{2}-1\right]$$

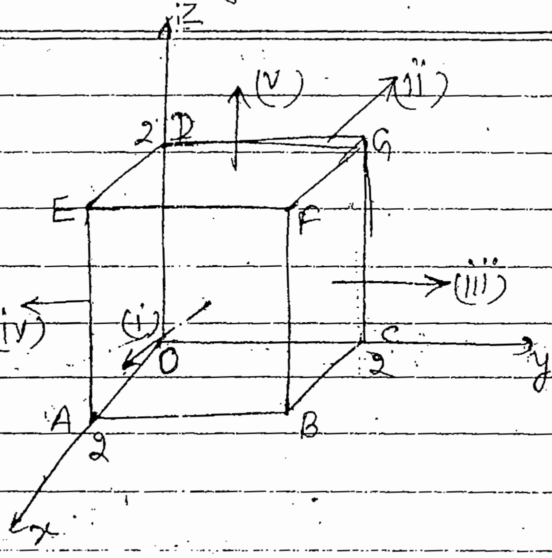
$$= y^2+2x\left(\frac{5}{2}\right) = y^2+5x \quad + 2x\left[\frac{y^2}{2}+y\right]_2^1 \quad -5x = 1+10-5 = \underline{\underline{6}}$$

Q.7: $\cancel{v = 2xz\hat{x}+(x+2)\hat{y}+y(z^2-3)\hat{z}}$

$$\oint_s \vec{v}\cdot d\vec{s} =$$

Q.7 :- $\int_S v \cdot ds = ?$

$$v = 2xz\,\hat{x} + (x+2)\hat{y} + y(z^3-3)\hat{z}$$



(i) $\iint_{ABEF} v \cdot ds = \int_0^2\int_0^2 v \cdot \hat{x}\, dy\, dz$

$$= \int_0^2\int_0^2 \left[2xz\,\hat{x} + (x+2)\hat{y} + y(z^3-3)\hat{z}\right] \cdot \hat{x}\, dy\, dz$$

$$= \int_0^2\int_0^2 2xz\, dy\, dz = 4\int_0^2\int_0^2 z\, dy\, dz \qquad [x=2]$$

$$= 4\,[y]_0^2 \left[\frac{z^2}{2}\right]_0^2 = 4(2)\left(\frac{4}{2}\right) = 16$$

(ii) $\iint_{OCGD} v \cdot ds = \int_0^2\int_0^2 v \cdot (-\hat{x})\, dy\, dz = 0 \qquad [x=0]$

(iii) $\iint_{BCFG} v \cdot ds = \int_0^2\int_0^2 v \cdot \hat{y}\, dx\, dz = \int_0^2\int_0^2 (x+2)\, dx\, dz$

$$= \left[\frac{x^2}{2} + 2x\right]_0^2 [z]_0^2 = (2+4)(2) = 12$$

(iv) $\iint_{OAED} v \cdot ds = \int_0^2\int_0^2 v \cdot (-\hat{y})\, dx\, dz = \int_0^2\int_0^2 -(x+2)\, dx\, dz = -12$

(v) $\iint_{DEFG} v \cdot ds = \int_0^2\int_0^2 v \cdot \hat{z}\, dx\, dy = \int_0^2\int_0^2 y(z^3-3)\, dx\, dy$

$$= \cancel{8}\int_0^2\int_0^2 y\, dx\, dy = \cancel{8}\left[\frac{y^2}{2}\right]_0^2 [x]_0^2 \qquad [z=2]$$

$$= \cancel{8}\left(\frac{4}{2}\right)(2) = \cancel{20}\,4$$

$$\int_S v \cdot ds = 16 + 0 + 12 - 12 + \cancel{20}4 = 20$$

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Note → If line integral of any vector is path independent then this is conservative vector. Curl of a conservative vector always zero. $\boxed{\vec{\nabla}\times\vec{V}=0}$

Q.9:- (a) $v = x^2\hat{x} + 2yz\,\hat{y} + y^2\hat{z}$

(a) $(0,0,0) \rightarrow (1,0,0) \rightarrow (1,1,0) \rightarrow (1,1,1)$

$$\int v\cdot dl = \int_{(0,0,0)}^{(1,0,0)} v\cdot dl \;(dx\,\hat{x}) + \int_{(1,0,0)}^{(1,1,0)} v\cdot dl \;(dy\,\hat{y}) + \int_{(1,1,0)}^{(1,1,1)} v\cdot dl \;(dz\,\hat{z})$$

$$= \int_0^1 x^2\,dx + \int_0^1 0\,dy + \int_0^1 1\,dz$$

$$= \left[\frac{x^3}{3}\right]_0^1 + [z]_0^1 = \frac{1}{3} + 1 = \frac{4}{3}$$

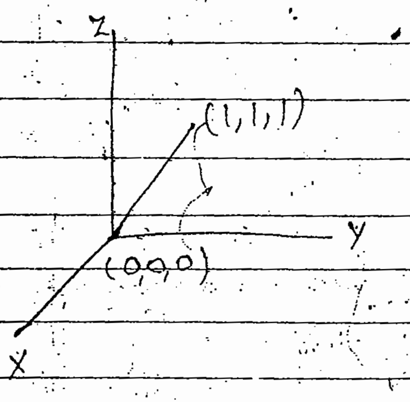
(b) $(0,0,0) \rightarrow (0,0,1) \rightarrow (0,1,1) \rightarrow (1,1,1)$

$$\int v\cdot dl = \int_0^1 v\cdot dz\,\hat{z} + \int_0^1 v\cdot \hat{y}\,dy + \int_0^1 v\cdot \hat{x}\,dx$$

$$= \int_0^1 y^2\,dz + \int_0^1 2yz\,dy_{\,(z=1)} + \int_0^1 x^2\,dx$$

$$= 0 + 2\left[\frac{y^2}{2}\right]_0^1 + \left[\frac{x^3}{3}\right]_0^1 = 1 + \frac{1}{3} = \frac{4}{3}$$

(c) The direct straight line
$(0,0,0) \rightarrow (1,1,1)$



Let the parametric eqn's to the straight line be
$x = t,\ y = t,\ z = t$
$x = y = z = t$

$V = t^2\,\hat{x} + 2t^2\,\hat{y} + t^2\,\hat{z}$

$r = t\,\hat{x} + t\,\hat{y} + t\,\hat{z}$

$dr = dt\,\hat{x} + dt\,\hat{y} + dt\,\hat{z}$

$$\int_c V\cdot dr = \int_{t=0}^{t=1} (t^2 + 2t^2 + t^2)\,dt = \int_0^1 4t^2\,dt$$

$$= 4\left[\frac{t^3}{3}\right]_0^1 = \frac{4}{3}$$

Here Line integral is independent of path so it is conservative.

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Q.10 :- $T = xy^2$

$a = (0,0,0)$, $b = (2,1,0)$

$$\int_a^b (\nabla T)\cdot d\bar{r} = T(b) - T(a)$$

$$\bar{\nabla} T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

$$\bar{\nabla} T = y^2 \hat{x} + 2xy\,\hat{y}$$

$$dr = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$(\bar{\nabla} T)\cdot dr = y^2 dx + 2xy\,dy$$

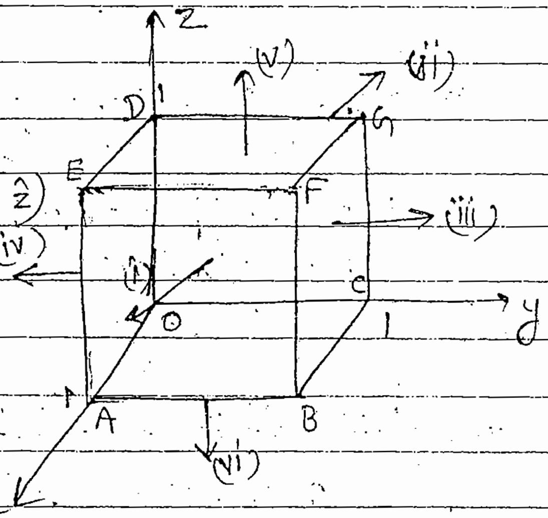
$$\int_a^b (\bar{\nabla} T)\cdot dr = \int_0^2 1\cdot dx + \int_0^1 2\cdot 2y\,dy$$

$$= [x]_0^2 + 4\left[\frac{y^2}{2}\right]_0^1 = 2 + 2 = 4$$

$$T(b) - T(a) = |xy^2|_{(2,1,0)} - |xy^2|_{(0,0,0)} = 2 - 0 = 2$$

Q.12 :- $\bar{v} = y^2\hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}$

Div Theorem $\iint_S \bar{A}\cdot dS = \iiint_V \text{div}\,\bar{A}\,dV$



$$\text{div}\,\bar{v} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{k}\right)\cdot$$

$$(y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\,\hat{z})$$

$$= 0 + 2x + 2y \quad 2(x+y)$$

$$\iiint_V \text{div}\,\bar{V}\,dV = \int_{z=0}^1\int_{y=0}^1\int_{x=0}^1 (2x+2y)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 [2xz + 2yz]_0^1\,dx\,dy$$

$$= \int_0^1\int_0^1 (2x + 2y)\,dx\,dy = \int_0^1 \left(2xy + 2\frac{y^2}{2}\right)_0^1 dx$$

$$= \int_0^1 (2x + 1)\,dx = 2\left[\frac{x^2}{2}\right]_0^1 + [x]_0^1 = 2$$

Cube has 6 faces.

(i) $$\iint_{OCGD} V\cdot dS = \int_0^1\int_0^1 V\cdot(-\hat{x})\,dy\,dz$$

$$= \int_0^1\int_0^1 -y^2\,dy\,dz = -\left[\frac{y^3}{3}\right]_0^1 [z]_0^1 = -\frac{1}{3}$$

(ii) $$\iint_{ABFE} V\cdot dS = \int_0^1\int_0^1 V\cdot\hat{x}\,dy\,dz$$

$$= \int_0^1\int_0^1 y^2\,dy\,dz = \frac{1}{3}$$

(iii) $$\iint_{OAED} V\cdot dS = \int_0^1\int_0^1 V\cdot(-\hat{y})\,dx\,dz$$

$$= \int_0^1\int_0^1 -(2xy+z^2)\,dx\,dz = -\int_0^1\int_0^1 (\cancel{2x}+z^2)\,dx\,dz$$

$$\cancel{= -\int_0^1\left[2xz+\frac{z^3}{3}\right]_0^1 dx = -\int_0^1\left(2x+\frac{1}{3}\right)dx}$$

$$\cancel{= -\left[2\frac{x^2}{2}+\frac{1}{3}x\right]_0^1 = -\left[1+\frac{1}{3}\right] = -\frac{4}{3}}$$

$$= -\left[\frac{z^3}{3}\right]_0^1[x]_0^1 = -\frac{1}{3}$$

(iv) $$\iint_{BCGF} V\cdot dS = \int_0^1\int_0^1 V\cdot\hat{y}\,dx\,dz$$

$$= \int_0^1\int_0^1 (2xy+z^2)\,dx\,dz = \frac{4}{3}$$

(v) $$\iint_{OABC} V\cdot dS = \int_0^1\int_0^1 V\cdot(-\hat{z})\,dx\,dy$$

$$= -\int_0^1\int_0^1 2yz\,dx\,dy = 0$$

(vi) $$\iint_{EFGD} V\cdot dS = \int_0^1\int_0^1 V\cdot\hat{z}\,dx\,dy = \int_0^1\int_0^1 2yz\,dx\,dy$$

$$= \int_0^1\int_0^1 2y\,dx\,dy = 2\left[\frac{y^2}{2}\right]_0^1[x]_0^1$$

$$= 1$$

Total $$\iint_S V\cdot dS = \cancel{\frac{1}{3}} - \cancel{\frac{1}{3}} - \frac{1}{3} + \frac{4}{3} + 0 + 1 = +1+1 = 2$$

$$\iint_S V\cdot dS = \iiint_V \operatorname{div} V\,dV \qquad \underline{\text{Varified}}$$

$\int_{OA} v \cdot \ldots \quad \int_0 v \cdot a y y = \int_0 (3y^2)\,dy = \frac{y^3}{3}\Big|_0^1 =$

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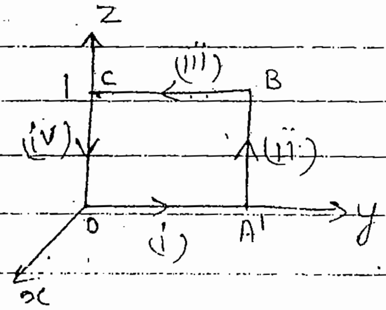
Q.13 :- $\bar{V} = (2xz + 3y^2)\hat{y} + (4yz^2)\hat{z}$

~~Stoke's theorem,~~

$$\int_C V \cdot dr = \iint_S \text{curl}\, V \cdot ds$$



$$\int_C V \cdot dr = \int_{OA} V \cdot dr + \int_{AB} V \cdot dr + \int_{BC} V \cdot dr + \int_{CO} V \cdot dr$$

In yz plane $x = 0$

$$\bar{V} = 3y^2\,\hat{y} + 4yz^2\,\hat{z}$$

$$\int_{OA} v \cdot dl = \int_{(0,0)}^{(1,0)} (3y^2\,dy + 4yz^2\,dz)$$

$$= [y^3]_0^1 + 4y\left[\frac{z^3}{3}\right]_0^0 = 1$$

$$\int_{AB} v \cdot dl = \int_{(1,0)}^{(1,1)} (3y^2\,dy + 4yz^2\,dz)$$

$$= [y^3]_1^1 + 4y\left[\frac{z^3}{3}\right]_0^1 = (1-1) + \frac{4}{3} = \frac{4}{3}$$

$$\int_{BC} v \cdot dl = \int_{(1,1)}^{(0,1)} (3y^2\,dy + 4yz^2\,dz)$$

$$= [y^3]_1^0 + 4y\left[\frac{z^3}{3}\right]_1^1 = -1$$

$$\int_{CO} v \cdot dl = \int_{(0,1)}^{(0,0)} (3y^2\,dy + 4yz^2\,dz)$$

$$= [y^3]_0^0 + 4 \times 0 \times \left[\frac{z^3}{3}\right]_1^0 = 0$$

Total

$$\int_C v \cdot dr = 1 + \frac{4}{3} - 1 + 0 = \frac{4}{3}$$

$$\text{curl}\,\bar{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 3y^2 & 4yz^2 \end{vmatrix}$$

$= \hat{x}[4z^2] - \hat{y}[0] + \hat{z}[0] = 4z^2\hat{x}$

$\iint_S \text{curl } V \cdot ds = \iint_S 4z^2\hat{x} \cdot dy\,dz\,\hat{x} = \iint_S 4z^2\,dy\,dz$

$= \iint_0^1 4\left[\frac{z^3}{3}\right]_0^1 [y]_0^1 = \frac{4}{3}$ Varified

Q.13 - $\int_0^\infty x e^{-x}\,dx$

$= -\left[x e^{-x}\right]_0^\infty + \int_0^\infty e^{-x}\,dx$

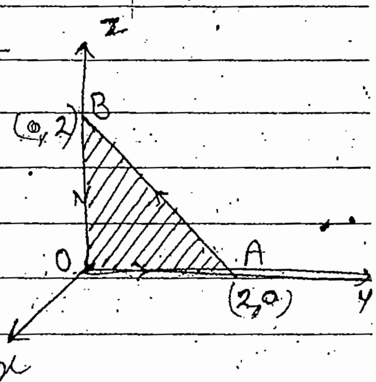
$= -[0-0] + \left[-e^{-x}\right]_0^\infty$

$= -\left[e^{-\infty} - e^{-0}\right] = -[0-1] = 1$ Ans

Q.14 - $\bar{V} = (xy)\hat{x} + (2yz)\hat{y} + (3zx)\hat{z}$

Stoks theorm

$\int_c V \cdot dl = \iint_S \text{curl } V \cdot ds$



$\int_c V \cdot dl = \int_{OA} V \cdot dl + \int_{AB} V \cdot dl + \int_{BO} V \cdot dl$

In yz plane $x = 0$

$\bar{V} = 2yz\,\hat{y} + 3zx\,\hat{z}$, $dr = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{z}$

$\int_{OA} V \cdot dr = \int_{(0,0)}^{(2,0)} 2yz\,dy = 0$

$y = 2 - z$, $dy = dz$

$\int_0^2 2(2-z)z\,dz = \int_0^2 (2z-4)\,dz$

$\int_{AB} V \cdot dr = \int_{(2,0)}^{(0,2)} 2yz\,dy = \cancel{4\int_{(2,0)}^{(0,2)} y\,dy = 4\left[\frac{y^2}{2}\right]_2^0}$

$= -8$

$(z^2 - 4z)_0^2 = 4 - 8$

$\int_{BO} V \cdot dr = \int_{(0,2)}^{(0,0)} 2yz\,dy = 0$

Total $\int_c V \cdot dr = 0 - 8 + 0 = -8$

$y = 2 - z, \; dy = -dz$

$\int_{AB} V\cdot dr = \int_2 2yz\,dy = \int_0^2 2(2-z)z\,(-dz) = -2\int_0^2 (2z-z^2)\,dz$

$= -2\left[2\frac{z^2}{2} - \frac{z^3}{3}\right]_0^2 = -2\left[4-\frac{8}{3}\right] = -2\left(\frac{4}{3}\right) = -\frac{8}{3}$

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Total $\int v\cdot dr = -\frac{8}{3}$

$$\text{Curl } \bar{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} = \hat{x}[-2y] - \hat{y}[3z-0] + \hat{z}[-x]$$

$$= -2y\,\hat{x} - 3z\,\hat{y} - x\,\hat{z}$$

$$\iint_S \text{curl } \bar{V}\cdot d\bar{s} = \iint_S (-2y\hat{x} - 3z\hat{y} - x\hat{z})\cdot dy\,dz\,\hat{x}$$

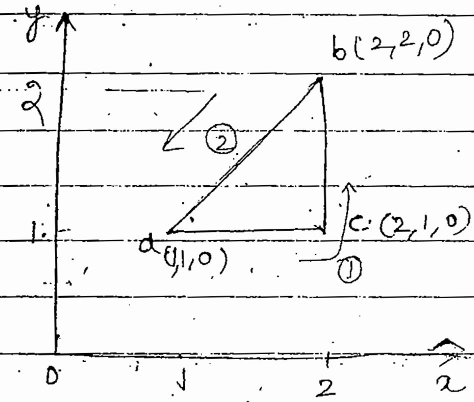
$$= \iint_S -2y\,dy\,dz = -2\int_0^{2}\int_0^{(2-z)} y\,dy\,dz$$

$$= -2\int_0^{2}\left(\frac{y^2}{2}\right)_0^{2-z} dz = -\int_0^{2}(2-z)^2\,dz \;\; \cancel{= -2\left[\frac{y^2}{2}\right]_0^{2}[z]_0^{(2-z)} = -8}$$

$$= -\left(4z + \frac{z^3}{3} - \frac{4z^2}{2}\right)_0^2 = -8/3$$

So $\int_C V\cdot dr = \iint_S \text{curl } \bar{V}\cdot ds$ Verified

Q.61 — $\bar{V} = y^2\hat{x} + 2x(y+1)\hat{y}$



$\oint \bar{v}\cdot d\bar{l} = \int_1 \bar{v}\cdot dl + \int_2 \bar{v}\cdot dl$

along ac, $d\vec{l} = dx\,\hat{x}$

$\int_{ac} \bar{v}\cdot d\bar{l} = \int_{ac} y^2\,dx \qquad (y=1)$

$= 1\int_1^2 dx = 1$ — ①

along cb,

$\int_{cb} \bar{v}\cdot d\bar{l} = \int_{cb} 2x(y+1)\,dy = 4\int_1^2 (y+1)\,dy \qquad (x=2)$

$= 4\left[\frac{y^2}{2} + y\right]_1^2 = 4\left[\frac{4}{2} + 2 - \frac{1}{2} - 1\right] = 4\times\frac{5}{2}$

$= 10$

So $\int_1 \bar{v}\cdot d\bar{l} = 1 + 10 = 11$

along ba, $d\bar{l} = dx\,\hat{x} + dy\,\hat{y}$

$\int_{ba} \bar{v}\cdot d\bar{l} = \int_{ba}\boxed{y^2\,dx + 2x(y+1)\,dy} \;\; \cancel{= \int_2^1 y^2\,dx + \int}$

choose a line eqn $y = x \Rightarrow dy = dx$

$= \int_2^1 (x^2 + 2x^2 + 2x)\,dx = 3\left[\frac{x^3}{3}\right]_2^1 + 2\left[\frac{x^2}{2}\right]_2^1 = -10$

$\oint \bar{v}\cdot dl = 11 - 10 = 1$ Ans

Application → $V = -\int \vec{E} \cdot dV$

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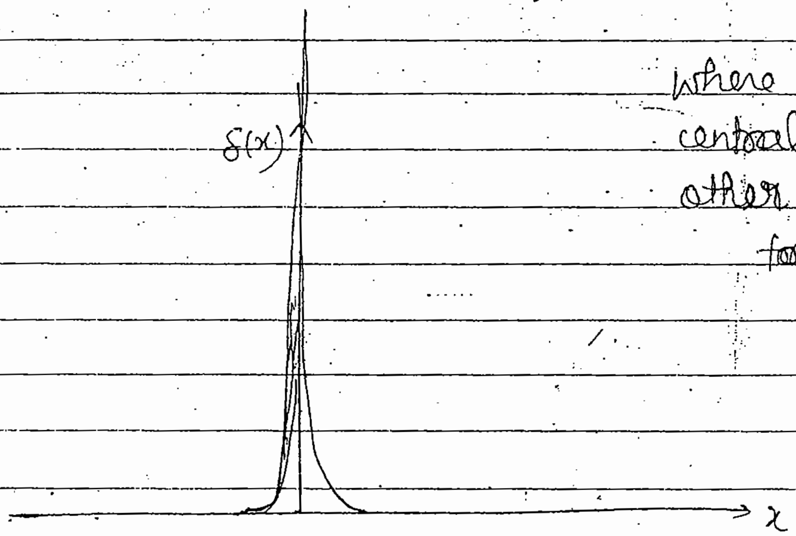
Dirac - Delta function :- This is not a ordinary fuc but generalized fucⁿ or distributed fucⁿ. It has the property that it is ∞ when x=

i.e $\delta(x) = \infty \quad x=0$

$\quad\quad = 0 \quad x \neq 0$

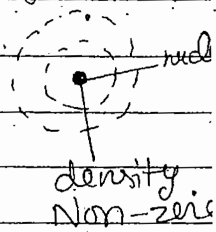
dirac-delta fucⁿ in 1 Dim

$x \rightarrow$ centre of the dirac delta fucⁿ.



where this fucⁿ is centralize it is ∞, at other points it is o

for ex -



Physical ex - density

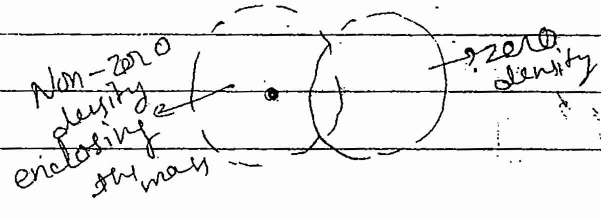
$$\vec{A} = \frac{\hat{r}}{r^2} = \frac{\vec{r}}{r^3}$$

$$\boxed{\vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi \delta^3(r)}$$

~~If we integrate~~

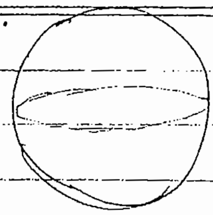
$$4\pi \int_{-1}^{1} \delta^3(r)\, d\tau \neq 0 \quad \& \quad 4\pi \int_{1}^{2} \delta^3(r)\, d\tau = 0$$

This integral will be non-zero when limits of r enclose it.



Divergence theorem

Q. $\oint_S \bar{A}\cdot d\bar{S} = \int_V (\bar{\nabla}\cdot\bar{A})\, d\tau$



We have a sphere of radius R.
Calculate L.H.S. & R.H.S.

Sol^n A·ds will be non-zero if ds is along $\hat{r}$ dir^n.

$ds = r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$ $\quad A = \frac{\hat{r}}{r^2}$

$$L.H.S. = \int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi = \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= \left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi}$$

$$= 4\pi$$

R.H.S. =

Mathematically $\bar{\nabla}\cdot\bar{A} = 0$
but physically it is conditionally 0, not always

$$R.H.S. = \int_V 4\pi\, \delta^3(r)\, d\tau$$

$$= 4\pi \int_V \delta^3(r)\, d\tau$$

$$= 4\pi$$

[integral = 1 if sphere is enclosing $r = 0$]
i.e. limits of r enclosing the volume.

Properties :-

* for 1 Dim

1. $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$

2. $f(x)\,\delta(x) = f(0)\,\delta(x)$ $\quad$ [x = 0 centre]

(f(x) — ordinary fun^n)

3. $\int_{-\infty}^{\infty} f(x)\,\delta(x)\, dx = f(0)\int_{-\infty}^{\infty}\delta(x)\, dx = f(0)$

Limits must enclose the origin of dirac delta fun^n.

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* If centre at $x = a$ then

$$\delta(x-a) = \infty \qquad x = a$$
$$= 0 \qquad x \neq a$$

1. $\int_{-\infty}^{\infty} \delta(x-a)\, dx = 1$

e.g. $\delta(x-2)$ centre at $x = 2$

$$\int_{3}^{4} \delta(x-2)\, dx = 0$$

because 2 is not in b/w 3 & 4 i.e. limits are not enclosing $x = 2$.

2. $f(x)\, \delta(x-a) = f(a)\, \delta(x-a)$

3. $\int_{-\infty}^{\infty} f(x)\, \delta(x-a)\, dx = f(a) \int_{-\infty}^{\infty} \delta(x-a)\, dx$

$$= f(a)$$

* Area of Dirac-Delta func^n :- is 1.

(height x width = finite)

* If $\boxed{\delta(Kx) = \frac{1}{|K|}\, \delta(x)}$ where K is constant

e.g. $\delta(-3x) = \frac{1}{3}\, \delta(x)$

* Dirac delta func^n is even func^n i.e.

$$\boxed{\delta(-x) = \delta(x)}$$

* Derivative of dirac delta func^n

$$\frac{d}{dx}\, \delta(x) = \delta'(x) = \frac{-1}{x}\, \delta(x)$$

* If there is a polynoimal in dirac delta fuc^n as
$\delta[x^2+2x+1]$.
Find find the roots $x = -1, -1$

~~If~~ $\delta[(x-a)(x-b)]$ is a dirac delta fuc^n who has two centres then

Imp. ① $\delta[(x-a)(x-b)] = \frac{1}{|a-b|}[\delta(x-a)+\delta(x-b)]$, $a \neq b$

No formula for common centre.

② $\delta[x^2-a^2] = \frac{1}{2|a|}[\delta(x-a)+\delta(x+a)]$, $a \neq 0$

* for nth derivative of dirac-delta fuc^n

$$\int_{-\infty}^{\infty} f(x)\, \delta^n(x)\, dx = (-1)^n f^n(0)$$

e.g. $\int_{-\infty}^{\infty} x^2 \delta'(x)\, dx = (-1)^1 \cancel{f'} 2x$
$= 0$

3-Dimensional Dirac Delta fuc^n :-

spherical-polar $\delta^3(r)$
Cartesian $\delta(x)\,\delta(y)\,\delta(z)$

Properties :-

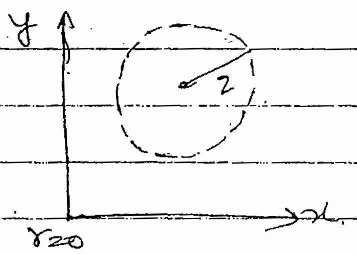
① $\int_{\text{all space}} \delta^3(r)\, dr = 1$

If limit of r enclosing the $r=0$ then its value is 1 otherwise 0.

e.g. for this sphere

$$\int_{\text{all space}} \delta^3(r)\, d\tau = 0$$

bcoz sphere is not enclosing centre at $r=0$.



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② $\boxed{\bar{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(r)}$

③ $f(r)\,\delta^3(r) = f(0)\,\delta^3(r)$

Note: $\bar{\nabla}\cdot E = \frac{\rho}{\epsilon_0}$

$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$

$\left\{ \begin{array}{l} \text{In } \nabla\cdot\vec{E} \text{ we use } \bar{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(r) \\ \& \quad \delta^3(r) = 1 \quad r=0 \\ \qquad\quad = 0 \quad r\neq 0 \end{array} \right\}$

Questions :-

① $\int_2^6 (3x^2-2x-1)\,\delta(x-3)\,dx$

② $\int_0^3 x^3\,\delta(x+1)\,dx$

③ $\int_0^2 (x^3+3x+2)\,\delta(1-x)\,dx$

④ $\int_{-1}^1 9x^2\,\delta(3x+1)\,dx$

⑤ $\int_{-1}^3 (x^3+2)\,\delta(x^2+5x+4)\,dx$

① $\int_2^6 (3x^2-2x-1)\,\delta(x-3)\,dx$

centre $a = 3$

$= (3(3)^2-2(3)-1)\int_2^6 \delta(x-3)\,dx$

limits enclosing $x=3$

So $\int_2^6 \delta(x-3)\,dx = 1$

$= 27-6-1$

$= 20$

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(2) $\int_0^3 x^3 \delta(x+1)\, dx$

centre $x = -1$

$= (-1)^3 \int_0^3 \delta(x+1)\, dx$

$= 0$

(3) $\int_0^2 (x^3+3x+2)\, \delta(1-x)\, dx$

centre $x = 1$

$= (1^3 + 3\cdot 1 + 2) \int_0^2 \delta(1-x)\, dx$, limits enclosing the centre $x=1$.

$= 6\,[1] = 6.$

(4) $\int_{-1}^{1} 9x^2\, \delta(3x+1)\, dx$

$\int_{-1}^{1} 9x^3\, \delta\left[3\left(x+\frac{1}{3}\right)\right] dx$

$\delta\left[3\left(x+\frac{1}{3}\right)\right] = \frac{1}{3}\,\delta\left(x+\frac{1}{3}\right)$

Centre $x = -\frac{1}{3}$

$\int_{-1}^{1} 9x^2\, \delta(3x+1)\, dx = \int_{-1}^{1} 9x^2\, \frac{1}{3}\, \delta\left(x+\frac{1}{3}\right) dx$

$= 3 \left(\frac{1}{-3}\right)^2 [1] = \frac{1}{3}$ Ans

(5) $\int_{-1}^{3} (x^3+2)\, \delta(x^2+5x+4)\, dx$

$\delta[x^2+5x+4] = \delta[(x+4)(x+1)]$

$a=4, \; b=1$

$= \frac{1}{3}\left[\delta(x+4) + \delta(x+1)\right]$

$\int_{-1}^{3} (x^3+2)\, \delta(x^2+5x+4) = \frac{1}{3}\int_{-1}^{3} (x^3+2)\, \delta(x+4)\, dx + \frac{1}{3}\int_{-1}^{3} (x^3+2)\, \delta(x+1)$

centre $= -4$ centre $= -1$

$$= 0 + \frac{1}{3}\left[(-1)^3 + 2\right] \int_{-1}^{3} \delta(x+1)\, dx$$

$$= \frac{1}{3}(+1)\,[1] = \frac{1}{3} \text{ Ans}$$

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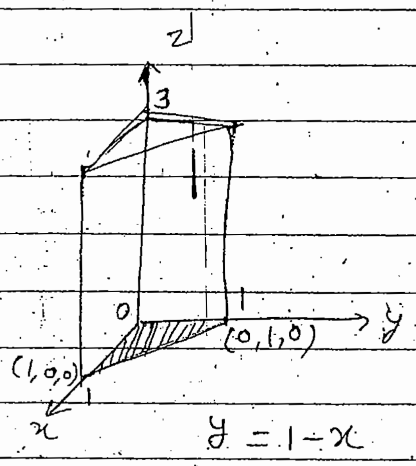
Q.8 :- $T = xyz^2$

volume $V = \int_V T\, d\tau$

$$V = \int_V xyz^2\, dx\, dy\, dz$$



$$= \int_0^3 z^2 dz \left[\int_0^1 x \left[\int_0^{1-x} y\, dy \right] dx \right]$$

$$= \left[\frac{z^3}{3}\right]_0^3 \left[\int_0^1 x \frac{(1-x)^2}{2}\, dx \right]$$

$$= \frac{27}{3}\frac{1}{2} \int_0^1 (x + x^3 - 2x^2)\, dx = \frac{9}{2}\left[\frac{x^2}{2} + \frac{x^4}{4} - \frac{2x^3}{3}\right]_0^1$$

$$= \frac{9}{2}\left[\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right] = \frac{9}{2}\left(\frac{6+3-8}{12}\right) = \frac{\cancel{9}^{3}}{2}\frac{1}{\cancel{12}_{4}}$$

$$= \frac{3}{8}$$

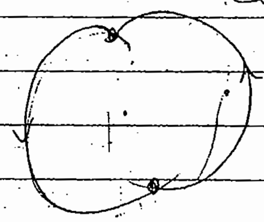
Note When work done by the force (line integral of force) is conservative.

$\boxed{F = -\vec{\nabla} V}$ conservative force

$$W = \int \overline{F} \cdot d\overline{l}$$

$\oint \overline{F} \cdot d\overline{l} = 0$ close line integral is always zero.

$\boxed{\nabla \times \overline{F} = 0}$, $\boxed{\nabla \times \overline{U} = 0}$



Q.10:- $T = xy^2$

$a = (0,0,0)$, $b = (2,1,0)$

gradient theorem $\int_a^b (\bar{\nabla} T) \cdot dr = T(b) - T(a)$

L.H.S. $= \int_a^b (\bar{\nabla} T) \cdot dr$

$\nabla T = \frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z}$

$= y^2 \hat{x} + 2xy \hat{y}$

$(\bar{\nabla} T) \cdot dr = y^2 dx + 2xy\, dy$

suppose $x = 2y \Rightarrow dx = 2\, dy$

$\int_a^b (\bar{\nabla} T) \cdot dr = \int_0^1 (2y^2 + 4y^2)\, dy$

$= 6 \int_0^1 y^2 dy = 6 \left[\frac{y^3}{3}\right]_0^1 = \frac{6}{3} = 2$

R.H.S. $= T(b) - T(a) = \left|xy^2\right|_{(2,1,0)} - \left|xy^2\right|_{(0,0,0)}$

$= 2 - 0 = 2$

L.H.S. = R.H.S. Ans

Q.11:- $T = x^2 + 4xy + 2yz^3$

$a = (0,0,0)$, $b = (1,1,1)$

(a) $(0,0,0) \rightarrow (1,0,0) \rightarrow (1,1,0) \rightarrow (1,1,1)$

Gradient Theorem $\int_a^b (\bar{\nabla} T) \cdot dr = T(b) - T(a)$

L.H.S $\int_a^b (\bar{\nabla} T) \cdot dr$

$\bar{\nabla} T = \frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z}$

$= (2x + 4y) \hat{x} + (4x + 2z^3) \hat{y} + 6yz^2 \hat{z}$

$(\bar{\nabla} T) \cdot dr = (2x + 4y) dx + (4x + 2z^3) dy + 6yz^2 dz$

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$$\int_a^b (\bar{\nabla}T)\cdot d\bar{r} = \int_0^1 (2x+4y)\,dx \underset{(y=0)}{} + \int_0^1 (4x+2z^3)\,dy \underset{(x=1,\, z=0)}{} + \int_0^1 6yz^2\,dz \underset{(y=1)}{}$$

$$= \int_0^1 2x\,dx + \int_0^1 4\,dy + \int_0^1 6z^2\,dz$$

$$= 2\left[\frac{x^2}{2}\right]_0^1 + 4\left[y\right]_0^1 + 6^2\left[\frac{z^3}{3}\right]_0^1$$

$$= 1 + 4 + 2 = 7$$

R.H.S. $= T(b) - T(a) = T(1,1,1) - T(0,0,0)$

$$= 7 - 0 = 7$$

L.H.S. = R.H.S. Varified

(b) $(0,0,0) \rightarrow (0,0,1) \rightarrow (0,1,1) \rightarrow (1,1,1)$

$$\int_a^b (\bar{\nabla}T)\cdot d\bar{r} = T(b) - T(a)$$

L.H.S. $= \int_a^b (\bar{\nabla}T)\cdot d\bar{r} = \int_0^1 6yz^2\,dz \underset{(y=0)}{} + \int_0^1 (4x+2z^3)\,dy \underset{(x=0,\, z=1)}{} + \int_0^1 (2x+4y)\,dx \underset{(y=1)}{}$

$$= \int_0^1 0\cdot dz + \int_0^1 2\,dy + \int_0^1 (2x+4)\,dx$$

$$= 0 + 2\left[y\right]_0^1 + 2\left[\frac{x^2}{2}\right]_0^1 + 4\left[x\right]_0^1$$

$$= 2 + 1 + 4 = 7$$

R.H.S. $= T(b) - T(a) = 7$

L.H.S. = R.H.S. Varified

(c) The parabolic path $z = x^2$, $y = x$

$$\int_a^b (\bar{\nabla}T)\cdot d\bar{r} = T(b) - T(a)$$

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L.H.S: $\int_a^b (\nabla T)\cdot d\vec{r} = \int_a^b \left[(2x+4y)dx + (4x+2z^3)dy + 6yz^2 dz\right]$

Put $z = x^2$ & $y = x \Rightarrow 2x\,dx = dz$, $dy = dz$

$$= \int_0^1 \left[(2x+4x)dx + (4x+2x^6)dx + \underbrace{6\,x\cdot x^4\cdot 2x\,dx}_{=12x^6}\right]$$

$$= \int_0^1 \left[10x + 14x^6\right]dx$$

$$= \cancel{10}^{5}\left[\frac{x^2}{\cancel{2}}\right]_0^1 + \cancel{14}^{2}\left[\frac{x^7}{\cancel{7}}\right]_0^1 = 5 + 2 = 7$$

$T(b) - T(a) = 7$

L.H.S. = R.H.S. Verified

Q.16: $\bar{V} = (r\cos\theta)\,\hat{r} + (r\sin\theta)\,\hat{\theta} + (r\sin\theta\cos\phi)\,\hat{\phi}$

$$\nabla\cdot\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, A_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_\phi$$

$$\nabla\cdot\bar{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\, r\cos\theta\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\, r\sin\theta\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[r\sin\theta\cos\phi\right]$$

$$= \frac{1}{r^2}3r^2\cos\theta + \frac{\cancel{r}}{\cancel{r}\sin\theta}2\sin\theta\cos\theta + - \sin\phi$$

$$= 3\cos\theta + 2\cos\theta + - \sin\phi$$

$$= 5\cos\theta - \sin\phi$$

div. theorem, $\oint_s V\cdot ds = \int_v \text{div}\,\bar{V}\,d\tau$

(for Hemisphere ~~of radius R~~ $\theta \to 0$ to $\pi/2$, $\phi \to 0$ to 2π)

$$\text{R.H.S.} = \int_0^R\int_0^{\pi/2}\int_0^{2\pi} (5\cos\theta - \sin\phi)\,\cancel{d\tau}\; r^2 dr\,\sin\theta\, d\theta\, d\phi$$

$$= \int_0^R r^2 dr \int_0^{\pi/2}\int_0^{2\pi}(5\cos\theta\sin\theta - \sin\phi\sin\theta)\,d\theta\,d\phi$$

$$= \left[\frac{r^3}{3}\right]_0^R \left[\int_0^{\pi/2}\int_0^{2\pi} 5\cos\theta\sin\theta\,d\theta\,d\phi - \int_0^{\pi/2}\int_0^{2\pi}\sin\phi\sin\theta\,d\theta\,d\phi\right]$$

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$$= \frac{R^3}{3}\left[5\int_0^{\pi/2}\frac{2\cos\theta\sin\theta}{2}\,d\theta\int_0^{2\pi}d\phi - \int_0^{\pi/2}\sin\theta\,d\theta\int_0^{2\pi}\sin\phi\,d\phi\right]$$

$$= \frac{R^3}{3}\left[\frac{5}{2}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}\left[\phi\right]_0^{2\pi} - \left[-\cos\theta\right]_0^{\pi/2}\left[-\cos\phi\right]_0^{2\pi}\right]$$

$$= \frac{R^3}{3}\left[-\frac{5}{4}(-1-1)(2\pi) - [0-1)(+1-1)\right]$$

$$= \frac{R^3}{3}\,\frac{5}{4}(2)(2\pi) = \frac{5\pi R^3}{3}$$

L.H.S. $\oint_S v\cdot ds = \iint(r\cos\theta\,\hat{r} + r\sin\theta\,\hat{\theta} + r\sin\theta\cos\phi\,\hat{\phi})\cdot(r^2\sin\theta\,d\theta\,d\phi\,\hat{r}$ $\cancel{+ r\,dr\,d\phi\,\hat{\theta} + r\,dr\,d\theta\,\hat{\phi})}$

$$\int_{curved} v\cdot ds_r + \int_{flat} \bar{v}\cdot ds = \iint(r^2\cos\theta\sin\theta\,d\theta\,d\phi + r^2\sin\theta\,dr\,d\phi + r^2\sin\theta\cos\phi\,dr\,d\theta)$$

$$= \left[\int_0^{\pi/2}\int_0^{2\pi}\frac{r^2}{2}\sin 2\theta\,d\theta\,d\phi + \int_0^{R}\int_0^{2\pi} r^2\sin\theta\,dr\,d\phi + \int_0^{R}\int_0^{\pi/2} r^2\sin\theta\cos\phi\,dr\,d\theta\right]$$

$$= \frac{r^2}{2}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}\left[\phi\right]_0^{2\pi} + \sin\theta\left[\frac{r^3}{3}\right]_0^{R}\left[\phi\right]_0^{2\pi} + \cos\phi\left[\frac{r^3}{3}\right]_0^{R}\left[-\cos\theta\right]_0^{\pi/2}$$

$$= -\frac{r^2}{4}(-1-1)2\pi + \sin\theta\,\frac{R^3}{3}\,2\pi \cancel{+} - \cos\phi\,\frac{R^3}{3}(0-1)$$

$$= \cancel{r^2\pi + \frac{2\pi}{3}R^3\sin\theta + \frac{R^3}{3}\cos\phi}$$

$$\int_{curved} v\cdot ds_r + \int_{flat}\bar{v}\cdot ds = \int_{curved} R\cos\theta\,R^2\sin\theta\,d\theta\,d\phi + \int_{flat} r\sin\theta\,r\,d\phi\,dr$$

$$= R^3\int_0^{2\pi}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\,d\phi + \int_0^{R}\int_0^{2\pi}\underbrace{\sin\theta}_{1}\,r^2\,d\phi\,dr \qquad \left\{\theta = \frac{\pi}{2}\right.$$

$$= \frac{R^3}{2}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}\left[\phi\right]_0^{2\pi} + \cancel{\sin\theta}\left(\frac{r^3}{3}\right)_0^{R}\left(\phi\right)_0^{2\pi}$$

$$= R^3\pi + \frac{2\pi R^3}{3} = \frac{5}{3}\pi R^3$$

L.H.S. = R.H.S. Varified

$\int_1 dy\,y + \int_2 dy\,y + dz\,z + \int_3 dz\,z$

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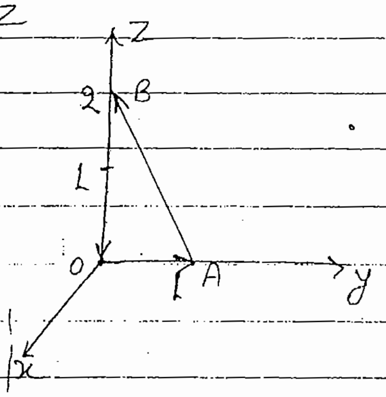
Q. 21 :- $\bar{V} = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$

Stok's Theorm,



$\int V\cdot dl = \int \text{curl } V \cdot dS$

$\cancel{\int_{OA} V\cdot dl = \int_{(0,1,0)}}$ In y-z plane

$x = 0$

$\vec{V} = yz^2\hat{y} + (3y+z)\hat{z}$

$\int V\cdot dl = \int_{AB} V\cdot dl + \int_{BO} V\cdot dl$

$= \int_{(0,1,0)}^{(0,0,2)} yz^2\,dy + \int_{(0,0,2)}^{(0,0,0)} (3y+z)\,dz \quad (y=0)$

$2y = 2 - z \Rightarrow z = 2 - 2y$

$y = \frac{2-z}{2}, \quad dy = \frac{1}{2}(-dz)$

$= \int_0^2 \left(\frac{2-z}{2}\right) z^2 \left(\frac{-dz}{2}\right) + \left[\frac{z^2}{2}\right]_2^0$

$= -\frac{1}{4}\int_0^2 (2z^2 - z^3)\,dz + \frac{1}{2}[-4]$

$= -\frac{1}{4}\left[\frac{2z^3}{3} - \frac{z^4}{4}\right]_0^2 - 2 = \frac{-1}{4}\left(\frac{16}{3} - \frac{16}{4}\right) - 2$

$= -\frac{1}{4}\cdot\frac{4}{3} - 2 = -\frac{1}{3} - 2 = -\frac{7}{3}$

$\text{curl } V = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & yz^2 & 3y+z \end{vmatrix}$

$= \hat{x}[3 - 2yz] - \hat{y}[0-0] + \hat{z}[0]$

$= (3 - 2yz)\hat{x}$

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$z(4+z^2-4z)$

$12-\frac{32}{3}$

$\frac{36-32}{3}$

R.H.S. $\int \text{curl} V \cdot dS = \iint (3-2yz)\, dy\, dz$

$$= \int_{z=0}^{2}\int_{y=0}^{\frac{2-z}{2}} (3-2yz)\, dy\, dz$$

$$= \int_0^2 3\,[y]_0^{\frac{2-z}{2}}\, dz - \int_0^2 2z\left[\frac{y^2}{2}\right]_0^{\frac{2-z}{2}} dz$$

$$= \int_0^2 3\left[\frac{2-z}{2}\right]dz - \int_0^2 z\left(\frac{2-z}{2}\right)^2 dz$$

$$= \int_0^2 \frac{3}{2}(2-z)\,dz - \int_0^2 \frac{1}{4}(4z+z^3-4z^2)\,dz$$

$$= \frac{3}{2}\left[2z-\frac{z^2}{2}\right]_0^2 - \frac{1}{4}\left[4\frac{z^2}{2}+\frac{z^4}{4}-4\frac{z^3}{3}\right]_0^2$$

$$= \frac{3}{2}\left[4-\frac{4}{2}\right] - \frac{1}{4}\left[4\cdot\frac{4}{2}+\frac{16}{4}-4\frac{8}{3}\right]$$

$$= \frac{3}{2}(2) - \frac{1}{4}\left[8+4-\frac{32}{3}\right] = 3-\frac{1}{4}\cdot\frac{4}{3} = 3-\frac{1}{3}$$

$$= +\frac{8}{3}$$

L.H.S. $= \int V\cdot dl = \int_{(0,1,0)}^{(0,0,2)} [yz^2\hat{y}+(3y+z)\hat{z}]\,[dy\,\hat{y}+dz\,\hat{z}]$

$$+ \int_{(0,0,2)}^{(0,0,0)} (3y+z)\,dz$$

Use $2y = y - z$

$y = \frac{2-z}{2}$, $dy = \frac{1}{2}(-dz)$

$$= -\frac{1}{3} + \int_{0,1,0}^{(0,0,2)} (3y+z)\,dz - 2$$

$$= -\frac{1}{3} + \int_0^2 \left[3\left(1-\frac{z}{2}\right)+z\right] dz - 2$$

$$= -\frac{1}{3}\left[3z-\frac{3}{2}\frac{z^2}{2}+\frac{z^2}{2}\right]_0^2 - 2$$

$$= -\frac{1}{3} + (6-3+2) - 2 = -\frac{1}{3}+5-2$$

$$= \frac{-1+15-6}{3} = \frac{8}{3}$$

L.H.S. = R.H.S.

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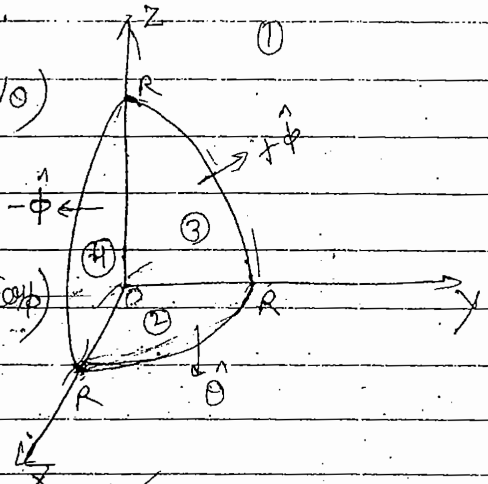
Q.20 :- $\bar{V} = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$

Div. Theorem, $\oint_s \bar{V}\cdot ds = \int_V \operatorname{div}\bar{V}\,d\tau$

This is the 1/8 part of sphere.



$\operatorname{div}\bar{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta V_\theta)$

$+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}V_\phi$

$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cdot r^2\cos\phi)$

$+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi)$

$= \frac{1}{r^2}4r^3\cos\theta + \cancel{\frac{r^2\cos\phi}{r\sin\theta}}\cos\theta \,\cancel{+}\, \frac{r^2\cos\theta}{r\sin\theta}(+\cos\phi)$

$= 4r\cos\theta$

$\oint_s \bar{V}\cdot d\bar{s} = \int_s [R^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}]\cdot[R^2\sin\theta\,d\theta\,d\phi\,\hat{r}$

$+ r\,dr\,d\phi\,\hat{\theta} + r\,dr\,d\theta\,\hat{\phi}]$

$= R^4\int_0^{\pi/2}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\,d\phi + \int_0\int_0^{R\;\pi/2} r^3\cos\phi\,dr\,d\phi -$

$\sin\phi\int\int r^3\cos\theta\,dr\,d\theta\,\}$

$= R^4\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2}[\phi]_0^{\pi/2} + \left[\frac{r^4}{4}\right]_0^R[\sin\phi]_0^{\pi/2}$

L.H.S.

$\oint\bar{v}\cdot d\bar{s} = \int_1\bar{V}\cdot d\bar{S}_r + \int_2\bar{V}\cdot d\bar{S}_\theta + \int_3\bar{v}\cdot d\bar{S}_\phi + \int_4\bar{v}\cdot(d\bar{S}_\phi)$

$d\bar{S}_r = R^2\sin\theta\,d\theta\,d\phi\,\hat{r}$

$\int_1\bar{V}\cdot d\bar{S}_r = R^4\int_0^{\pi/2}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\,d\phi$

$= R^4\frac{\pi}{2}\frac{1}{2}\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2} = -\frac{R^4\pi}{4}\frac{(-1-1)}{2} = \frac{R^4\pi}{4}$

$\int_2\bar{V}\cdot d\bar{S}_\theta = \int_0^R\int_0^{\pi/2} r^2\cos\phi\,r\,dr\,d\phi = \left[\frac{r^4}{4}\right]_0^R(\sin\phi)_0^{\pi/2}$

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$$= \frac{R^4}{4}(1-0) = \frac{R^4}{4}$$

along y-axis. $\phi = \pi/2$, $\sin\frac{\pi}{2} = 1$

$$\int_3 \bar{V}\cdot d\bar{S}_\phi = \int_0^R \int_0^{\pi/2} -r^2 \cos\theta \sin\phi \cdot r \, dr \, d\theta$$

$$= -\frac{R^4}{4}\left[\sin\theta\right]_0^{\pi/2} = -\frac{R^4}{4}$$

$$\int_4 \bar{V}\cdot d\bar{S}_\phi = 0$$ bcz $\phi = 0$ along x-axis i.e. $\phi = 0$ in x-z plan

$$\oint \bar{V}\cdot d\bar{S} = \frac{R^4\pi}{4} + \frac{R^4}{4} - \frac{R^4}{4} + 0 = \frac{\pi R^4}{4}$$

$$R.H.S = \int_V \operatorname{div}\bar{V}\, d\tau = \int_0^R\int_0^{\pi/2}\int_0^{\pi/2} 4r\cos\theta \, r^2 \, dr \sin\theta \, d\theta \, d\phi$$

$$= \frac{4}{2}\left[\frac{r^4}{4}\right]_0^R \left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2} \left[\phi\right]_0^{\pi/2}$$

$$= \frac{\cancel{2}}{\cancel{2}}\frac{R^4}{4}\left[-1-1\right)\frac{\pi}{2} = \frac{\pi R^4}{4}$$

Electrostatics

* Constant electric field which is not changing with time.

(i) $\vec{E} = r\cos\theta\,\hat{r}$

(ii) $\vec{E} = r\cos\omega t\,\hat{r}$

(iii) $\vec{E} = rt\,\hat{r}$

(iv) $\vec{E} = ct\,\hat{r}$

(v) $\vec{E} = c\,\hat{r}$

(vi) $\vec{E} \quad c\,\hat{\theta}$

Here (i), (v) & (vi) are constant electric field.

* If ~~space~~ electric field is not changing with space - uniform.
Changing with space - Non uniform.
$(x, y, z, r, \theta, \phi]$

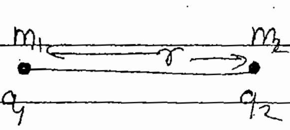
(i), (ii), (iii) → Non-uniform
(iv), (v), (vi) → Uniform

* If a fun^n is constant with space & time (x, y, z, t) then it will be uniform & constant.

Coulomb Force :-

4 fundamental force
EM (∞), gravitational (∞), strong, weak
→ Coulomb force comes under Electromag force.
Its range is ∞.

Note :- magnitude of gravitational force is very smaller than magnitude of EM force. That's why effect of gravitational force can be seen on massive bodies.



$$F_G \propto \frac{M_1 M_2}{r^2} \quad \& \quad F_c = \frac{q_1 q_2}{r^2}$$

Force → Vector Quantity

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$F_G = G\frac{m_1 m_2}{r^2}$ This is primarily attractive

$F_c = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$ It is attractive as well as repulsive depending on the nature of charge

If q_1, q_2 are of same sign then repulsive.
" " " opposite " " attractive.

Note:- Strong & weak forces are of short range, so these forces found in nucleus.

Energy quanta of EM force is photon. In b/w two charged bodies there will be exchange of photons. & energy quanta of gravitational force is graviton.

Ques 1- Two free charges $+4q$ & $+q$ are placed at a distance r apart. find the coulomb force b/w them. Also find magnitude, sign & location of 3rd charge q' such that

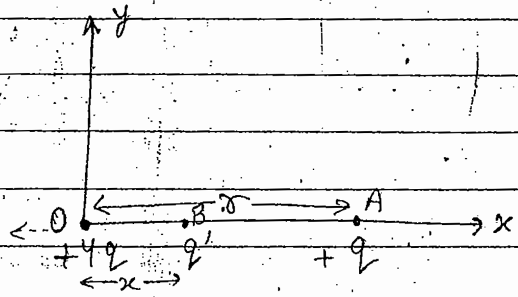
(i) No force on q'

(ii) No force on either of the charge $+4q$, $+q$ & q'

This is called Entire system is in equilibrium.

$$\vec{F}_{4q} = \frac{1}{4\pi\epsilon_0}\frac{4q\cdot q}{r^2}\hat{r}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{4q^2}{r^2}(-\hat{x})$$



$$\vec{F}_q = \frac{1}{4\pi\epsilon_0}\frac{4q^2}{r^2}(+\hat{x})$$

(i) Suppose $q' = +ve$

$$\vec{F}_q' = \frac{1}{4\pi\epsilon_0}\frac{4qq'}{x^2}(+\hat{x}) \quad \text{—(1)}$$

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$$\vec{F_{q'}} = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(r-x)^2}(-\hat{x}) \quad \text{---(2)}$$

There will be no force on q' if magnitude of these forces ((1) & (2)) are same bcz dir^n are already opposite.

Note :- If there are more than 1 force on a point then total force at this point will be vector sum of these forces. $\vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3} + \cdots \vec{F_n}$

(1) + (2)

$\Rightarrow$ $\vec{F_{q'}} + \vec{F_{q'}} = 0$

$$\frac{1}{4\pi\epsilon_0}\frac{4qq'}{x^2}\hat{x} = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(r-x)^2}(-\hat{x})$$

$\Rightarrow$ $4(r-x)^2 = x^2$

$4r^2 + 4x^2 - 8rx = x^2$

$3x^2 - 8rx + 4r^2 = 0$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{8r \pm \sqrt{64r^2 - 48r^2}}{6} = \frac{8r \pm 4r}{6}$$

$$= 2r \text{ or } \frac{2r}{3}$$

$x = 2r$ is not acceptable.

So $x = \frac{2r}{3}$ distance of q' from $+4q$.

i.e. if $x = \frac{2r}{3}$ then there will be no force on q'

Now if q' is -ve

then force b/w $4q$ & q' and also b/w q' & q is attractive.

$$\vec{F_{q'}} = \frac{1}{4\pi\epsilon_0}\frac{4qq'}{x^2}(-\hat{x})$$

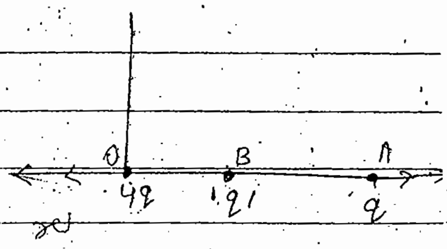
$$\vec{F_{q'}} = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(r-x)^2}(\hat{x})$$

In this case distance will be same

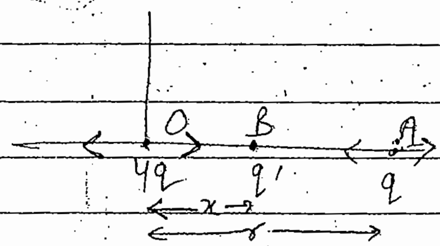
$$x = \frac{2r}{3}$$

Zero force condⁿ can be achieved by either +ve or −ve or any other magnitude charge.

(ii) If $q' = +ve$, equilibrium can not be achieved in entire system. So zero force condⁿ can not be achieved.



If $q' = -ve$ then zero force condⁿ can be achieved & entire system will be in equilibrium.



$$\vec{F}_{4q} = \frac{1}{4\pi\epsilon_0}\frac{4qq'}{x^2}(+\hat{x})$$

$$\vec{F}_{4q} = \frac{1}{4\pi\epsilon_0}\frac{4qq}{r^2}(-\hat{x})$$

$$\vec{F}_{4q} + \vec{F}_{4q} = 0$$

$$\frac{1}{4\pi\epsilon_0}\frac{4qq'}{x^2} = \frac{1}{4\pi\epsilon_0}\frac{4qq}{r^2}$$

$$\frac{q'}{x^2} = \frac{q}{r^2}$$

$$q' = \frac{q}{r^2} \times x^2 = \frac{q}{r^2} \times \frac{4r^2}{9}$$

$$\boxed{q' = -\frac{4q}{9}}$$

If we put q' at $\frac{4q}{9}$ distance then there will be no force on entire system.

~~Note~~ :- ~~If 4q~~ A

Note 1- If both charges are of same sign then ~~force b~~ equilibrium point (point at which net force $= 0$) is in ~~the~~ middle between these two charges.

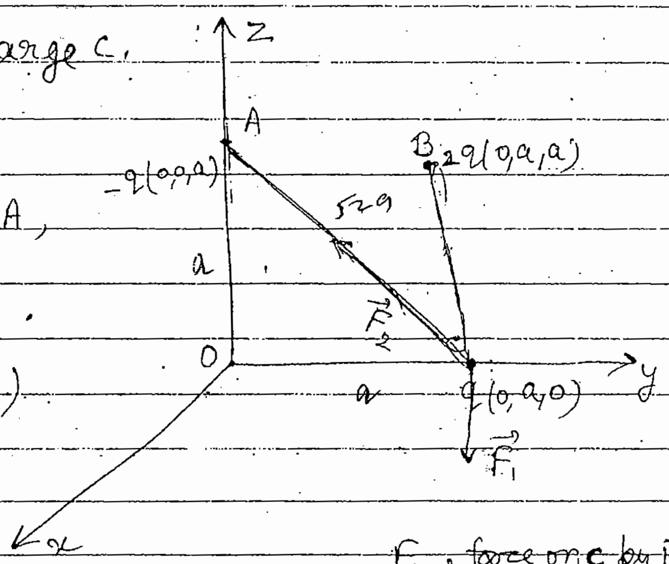
But if both charges are of opposite sign then equ. point is outside of this system.

i.e. Equilibrium point does not exist in the system for dissimilar charges.

Similar charges $\rightarrow$ within the system

dissimilar charges $\rightarrow$ outside the system

Que:- Find the force on charge c.



$\vec{F_c} = ?$

~~force on charge c by charge A,~~

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^3}\vec{r}$$

$$\vec{F_1} = \frac{1}{4\pi\epsilon_0}\frac{2qq}{a^2}(-\hat{z}) \quad \text{—(1)}$$

$$\vec{F_2} = \frac{1}{4\pi\epsilon_0}\frac{q^2(a\hat{z}-a\hat{y})}{(\sqrt{2}a)^3} \quad \left\{\begin{array}{l}\vec{r} = -a\hat{z}\\ |\vec{r}| = a\end{array}\right\}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q^2(\hat{z}-\hat{y})}{2\sqrt{2}a^2} \quad \left\{\begin{array}{l}\vec{r} = a\hat{z}-a\hat{y}\\ |\vec{r}| = \sqrt{2}a\end{array}\right\} \quad \text{—(2)}$$

$F_1 \rightarrow$ force on c by B

$F_2 \rightarrow$ force on c by A

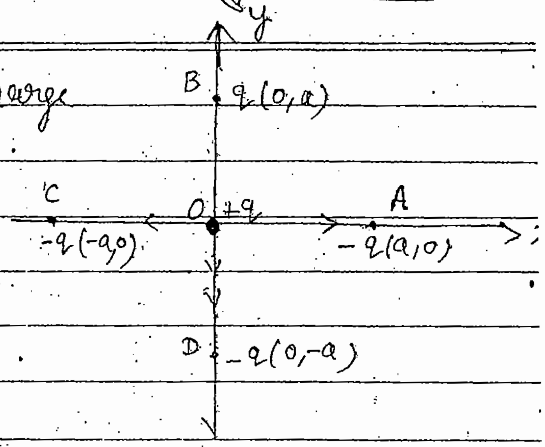
Resultant force on C is

$$\vec{F_c} = \vec{F_1} + \vec{F_2}$$

$$\boxed{\vec{F_c} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}\left[\left(-2+\frac{1}{2\sqrt{2}}\right)\hat{z} - \frac{1}{2\sqrt{2}}\hat{y}\right]}$$

This is the resultant force on C charge.

Ques:- Find the force on a fifth charge $+q$ placed at O.



$$\vec{F}_{OA} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^3} a\hat{x}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}\hat{x}$$

$$F_{OB} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}(-\hat{y})$$

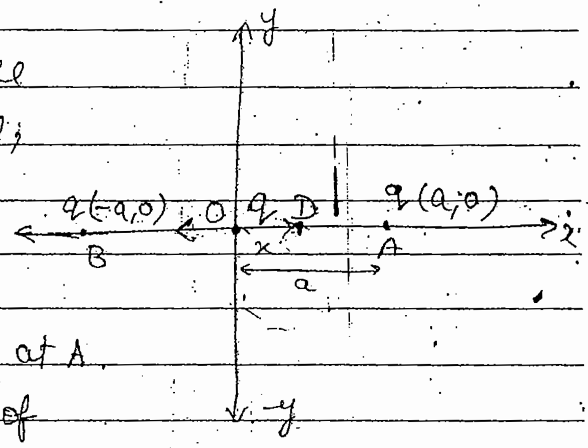
$$F_{OC} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}(-\hat{x})$$

$$F_{OD} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}(-\hat{y})$$

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}[-2\hat{y}] \Rightarrow \vec{F} = \frac{-1}{4\pi\epsilon_0}\frac{2q^2}{a^2}\hat{y}$$ Ans

Ques:- If a third charge $+q$ place at O, what is the force on the charge → zero.

If charge is displaced a small distance x towards the charge placed at A. Find the freq. of oscillation of the charge placed at O.



Note:- -ve charge will oscillate in b/w -ve charge &
+ve " " " " +ve "

oscillating charge is in stable equilibrium.
If -ve charge is placed b/w +ve charges then there will be unstable equilibrium.

In Simple harmonic motion:-

$$F \propto +x$$

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$$F = kx \quad \text{---(1)} \qquad \omega = \sqrt{\frac{K}{m}}$$

If the mass of $+q$ charge is m then resultant force on D,

$$\vec{F_D} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{(a-x)^2}(-\hat{x}) + \frac{1}{4\pi\epsilon_0}\frac{q^2}{(a+x)^2}(+\hat{x})$$

$$|\vec{F_D}| = \frac{-q^2}{4\pi\epsilon_0}\left[\frac{1}{(a-x)^2} - \frac{1}{(a+x)^2}\right]$$

$$= -\frac{q^2}{4\pi\epsilon_0}\left[\frac{(a+x)^2-(a-x)^2}{(a-x)^2(a+x)^2}\right]$$

$$= -\frac{q^2}{4\pi\epsilon_0}\left[\frac{4ax}{(a)^2(a)^2}\right] \qquad x << a$$

$$\boxed{|\vec{F_D}| = \frac{-q^2 x}{\pi\epsilon_0 a^3}} \quad \text{---(2)}$$

$$(1) \Rightarrow \quad +\frac{q^2 x}{\pi\epsilon_0 a^3} = Kx$$

$$\boxed{K = \frac{q^2}{\pi\epsilon_0 a^3}}$$

ang. freq. $\quad \omega = \sqrt{\frac{K}{m}} = \sqrt{\frac{q^2}{m\pi\epsilon_0 a^3}}$

$$\boxed{\omega = \sqrt{\frac{q^2}{m\pi\epsilon_0 a^3}}}$$

linear freq. $\quad f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{q^2}{m\pi\epsilon_0 a^3}}$

$$\boxed{f = \frac{q}{2\pi a}\sqrt{\frac{1}{m\pi\epsilon_0 a}}}$$

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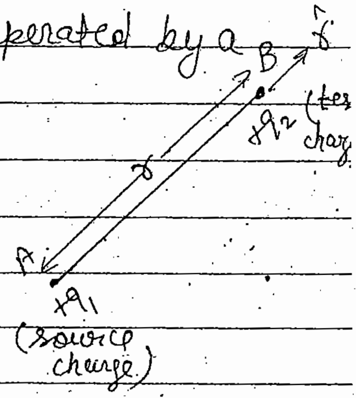
Electric Field :- It is defined as (coulomb) force per unit charge

$$\vec{E} = \frac{\vec{F}}{q}$$

Suppose we have 2 charge, $+q_1$ & $+q_2$ seperated by a distance r.



force on q_2 due to q_1

$$\vec{F}_B = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}$$

electric field at B,

$$\vec{E}_B = \frac{\vec{F}_B}{q_2}$$

souce charge → due to which Ele. field is find out

Test " → at which electric field is find out

$$\boxed{q_2 = +1\,C}$$

Test charge is always of unity magnitude & it is always +ve.

* Find Electric field at point B means find the force of test charge placed at B.

Direction of electric field depanand on source charge.

If source charge +ve → away

−ve → toward.

so $\boxed{\vec{F} = q\vec{E}}$

Electric Flux :- Flux = Electric field X area

$= \vec{E} \cdot \vec{A}$

$\vec{E} \cdot \vec{A}$ = Total no of electric field lines

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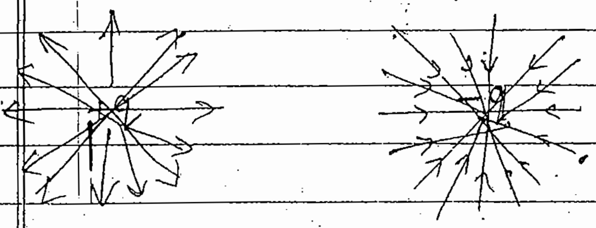
Electric field lines :- It is a hypothetical concept. To find out the strength of a charge we use electric field lines concept.

from +ve charge, electric field lines diverges out, -ve charge, lines are inward

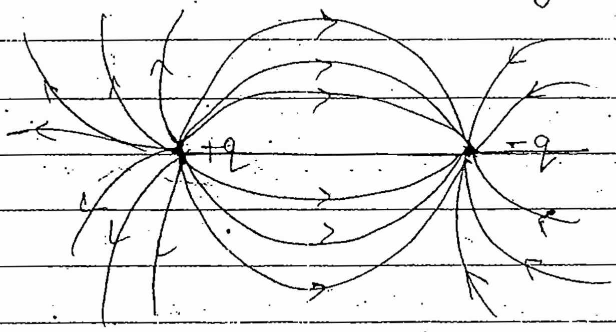


E field lines show the dir^n of electric field.

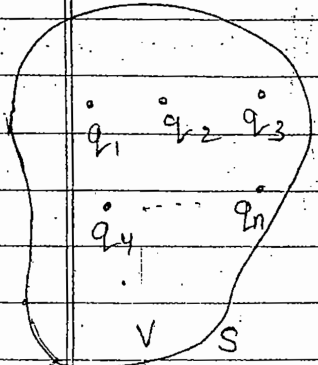
E field lines never crosses each other. If they cross, they show two dir^n.

E field lines are open curves.

If two charges are placed together, then E field lines originates at +ve charge & terminate at -ve charge.



Gauss Law :- Acc. to this law, Electric flux passing through a closed surface S (is equal to charge enclose in the surface devided by ϵ_0) is $1/\epsilon_0$ times charge enclose by the surface



$$\boxed{\oint \vec{E} \cdot d\vec{S} = \frac{q_{enc}}{\epsilon_0}}$$

q_{enc} → charge enclosed

This is Integral form of Gauss law.

Suppose we have no. of charges in the surface then flux passing through this surface. ~~This~~

$$\boxed{\oint \vec{E}\cdot d\vec{s} = \frac{\sum_{i=1}^{n} q_i}{\epsilon_0}}$$

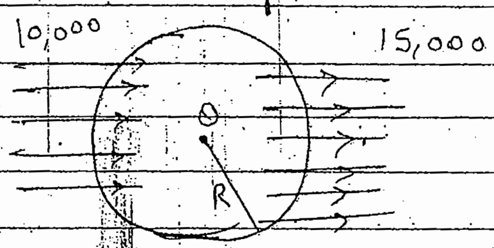
This flux is also known as ~~no~~ net no. of electric field lines contained in the surface.

If we have some +ve charge in closed surface the electric field lines go out.

This law is valid only for closed surface

e.g. Suppose we have a sphere of radius R in this we have 10,000 electric field lines entering the surface & 15,000 electric field lines leaving the surface. Calculate the charge contained in the surface.

Net lines = 15,000 − 10,000
= 5,000



So there is a source inside the surface.

$$\oint \vec{E}\cdot d\vec{s} = \frac{q_{enc}}{\epsilon_0}$$

$$q_{enc} = \epsilon_0 \times 5000$$
$$= 5000\,\epsilon_0 \quad \text{Ans}$$

* Gauss law is always valid but not useful, it is useful only if there is symmetry in the problem (it can be spherical symmetry, cylindrical symmetry or planer symmetry).

* Gauss law can not be used to calculate the electric field if monopole moment is zero. (monopole mom → total charge)

Gauss law can be used only if "Inverse

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square law" is valid.

$$E \propto \frac{1}{r^2}$$

$$F = qE \quad \text{so} \quad F \propto \frac{1}{r^2}$$

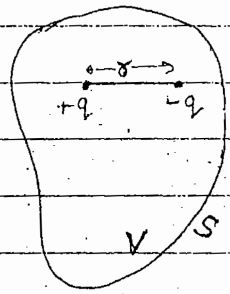
If we have a closed surface bounded volume V & there place two equal & opposite charges seperated by a finite distance r so

charge enclosed is $= 0$

Monopole moment $= 0$

$$\oint \vec{E}\cdot d\vec{S} = 0$$

$$\boxed{E_{mono} = 0}$$



Here dipole term will be non-zero, only monopole mom is zero.

→ Electric field can be calculated by Gauss law only if monopole mom. is ~~zero~~ non-zero.

for dipole, $E \propto \frac{1}{r^3}$

So Inverse square law is not valid.

→ If flux is 0. It is not necessaryly mean that electric field is also 0.

→ **Differential form of Gauss Law:-**

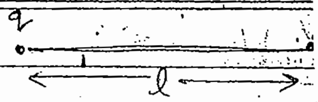
$$\oint_S \vec{E}\cdot d\vec{S} = \frac{q_{enc}}{\epsilon_0}$$

$$\Rightarrow \int_V (\vec{\nabla}\cdot\vec{E})\, d\tau = \frac{q_{enc}}{\epsilon_0} \qquad \text{——— (1)}$$

Charge Densities :-

(i) Line charge density (λ) :- If a charge is concentrated at a point then it is a point charge. If charge is uniformely ~~dis~~

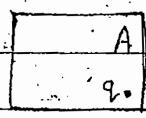
distributed over a line, then it is called line charge.



$$\lambda = \frac{q}{l}$$

$$\boxed{q = \int \lambda \, dl}$$

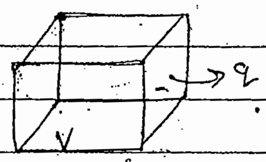
(ii) Surface charge density (σ) :- If charge is distributed over the surface then



$$\sigma = \frac{q}{A} \text{ or } \frac{q}{S}$$

$$\boxed{q = \int \sigma \, ds}$$

(iii) Volume charge density (ρ) :- If charge is distributed over the volume (can be cube or sphere or ...)



$$\rho = \frac{q}{V}$$

$$\boxed{q = \int \rho \, dV}$$

These are the 3 charge density → line, surface & volume.

So finally, Gauss law in diff. form is

① ⇒ $$\int_V (\nabla \cdot \vec{E}) \, d\tau = \frac{1}{\epsilon_0} \int \rho \, d\tau$$

$$\int_V \left(\nabla \cdot E - \rho/\epsilon_0\right) d\tau = 0$$

$$\boxed{\nabla \cdot E = \frac{\rho}{\epsilon_0}}$$

Volume is arbitrary, it can not be zero.

So $$\boxed{\nabla \cdot E = \frac{\rho}{\epsilon_0}}$$

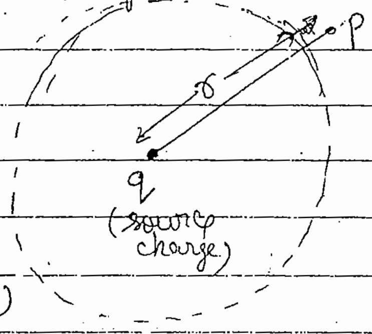
$\left[\begin{array}{l} \nabla \cdot D = \rho \\ \nabla \cdot E = \rho/\epsilon_0 \end{array} \right.$

later on it is known as Maxwell's eqn.

Problem :- Electric field of a point charge.

Suppose we have a point charge, we have to find electric field at a point P from the point charge q.

i.e. At P, a unity test charge is placed.

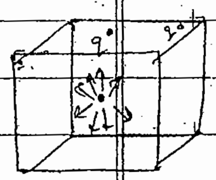


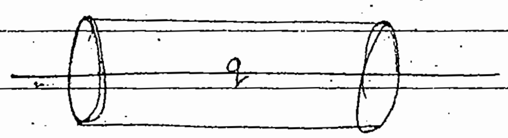
To find out flux, we need a closed surface (Gaussian surface)

→ We consider a sphere, becz each & every point of the spherical surface will be at same distace from the charge.

(for ex- If we take a cylender or cube then each point of cylender or cuble will not be at same distane from point charge).

→ [In case of line charge, we take cylender] i.e. cylendrical





Equation of the flux

$$\oint \vec{E}\cdot d\vec{s} = \frac{q_{enc}}{\epsilon_0}$$

for spherical surface $d\vec{S}_r = r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$

$$\Rightarrow E r^2 \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = \frac{q}{\epsilon_0}$$

$$\Rightarrow E r^2 [\cos\theta]_0^{\pi} [2\pi] = \frac{q}{\epsilon_0}$$

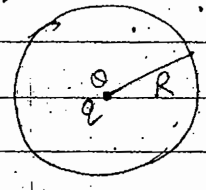
$$\boxed{\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}}$$

Electric field is radially outward.

Note :- If there is no func^n of θ & ϕ then
$$d\tau = 4\pi r^2 dr$$

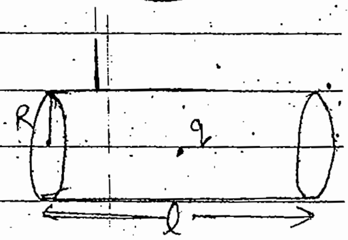
Problem Flux :-

If charge q is enclosed in a sphere of radius R, what is the flux passing through the surface.



Flux passing through the sphere $= \frac{q}{\epsilon_0}$

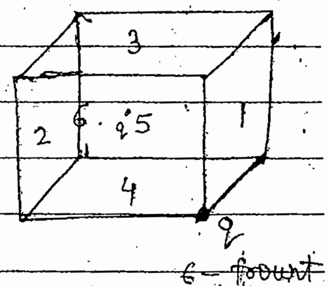
through cylinder $= \frac{q}{\epsilon_0}$



i.e. flux depends upon charge not on the surface.

→ If a charge q is placed in the cube, then total flux passing through the surface of cube $= \frac{q}{\epsilon_0}$ ~~each face~~

Flux passing through each face (6 faces are here)
$$= \frac{q}{6\epsilon_0}$$



(1) If charge placed at a corner, then flux pass through 1 surface $= 0$

4 $= 0$

6 $= 0$

This charge placed at corner is not enclosed. To cover this charge we need 8 cubes.

faces in one cube $= 6$

" 8 " $= 48$

Out of 48 faces from some face flux pass & from some face not pass.

The planes which are sharing the charge, from these planes no flux pass.

Surface 1, 4 & 6 share the charge so flux pass through these surface = 0

The flux is shared by 8 cubes i.e. $\frac{q}{\epsilon_0} \rightarrow 8$

So flux pass through the each single cube $= \frac{q}{8\epsilon_0}$

cube has 6 faces. Out of 6, ~~one~~ flux pass through only by 3 faces so flux pass through this cube $= \frac{q}{8\epsilon_0} \times \frac{1}{3}$

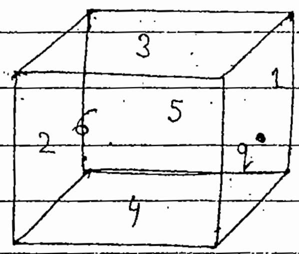
$= \frac{q}{24\epsilon_0}$

(2) Now,

charge is placed at face centred position.

To enclose the charge we need one more cube.

In two cubes → 12 faces



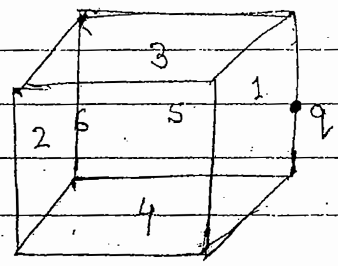
flux pass through this whole cube $= \frac{q}{2\epsilon_0}$

& Out of 6 faces, flux pass through 5 faces so flux pass through the cube $= \frac{q}{2\epsilon_0} \times \frac{1}{5}$

$= \frac{q}{10\epsilon_0}$

(3) To enclose charge we need 3 more cubes

$= \frac{q}{4\epsilon_0}$

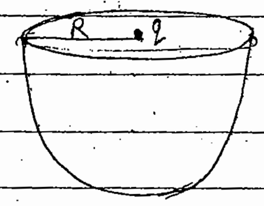


Out of 6 → flux pass through 4 ~~sur~~ faces

So flux $= \frac{q}{4\epsilon_0} \times \frac{1}{4} = \frac{q}{16\epsilon_0}$

(4) How much flux pass through the Hemispherical surface.

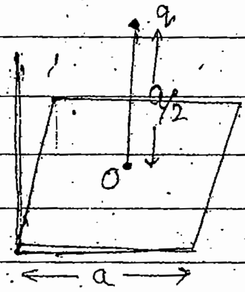
→ To cover this charge,
we need full sphere



flux $= \dfrac{q}{2\epsilon_0}$

(5) A charge is placed at $a/2$ distance from a surface.

To cover this charge we need to sketch a cube.



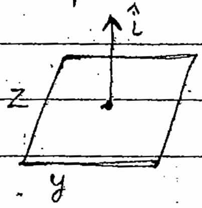
So flux $= \dfrac{q}{6\epsilon_0}$

Problem :- The Electric field in a region is given by

$$E = \frac{3}{5}E_0\hat{i} + \frac{4}{5}E_0\hat{j} + \frac{2}{3}E_0\hat{k}$$

where E_0 ~~as~~ $= 2\times10^3$ V/m

Find the flux of this field through a rectangular surface of area $0.2\ m^2$ parallel to y-z plane.



flux $\phi_E = \vec{E}\cdot\vec{A}$

$= \frac{3}{5}E_0 \times 0.2$

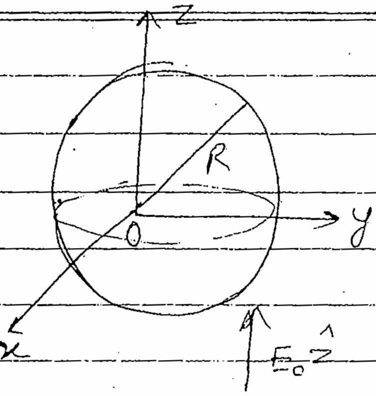
$= \frac{3}{5}\times 2\times10^3 \times 0.2$

$= \frac{12\times10^2}{5} = \frac{1200}{5}$

$$\boxed{\phi_E = 240\ \text{V-m}}$$

Mag. of electric field is not depending on space co-ordinate so field is uniform.

Prob 1 - Uniform Electric field $E_0 \hat{z}$ present in the space & a sphere of radius R is placed in that space. Find the flux passing through the sphere in space.

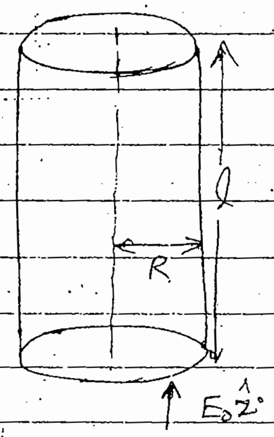


The flux entering inside the sphere is equal to the flux leaving the sphere so net flux is zero (passing through the sphere)

$$\boxed{\phi_E = \text{zero}}$$

Prob :- flux passing through the cylender of length l & radius R. Ele. field is uniform.

Net flux = 0



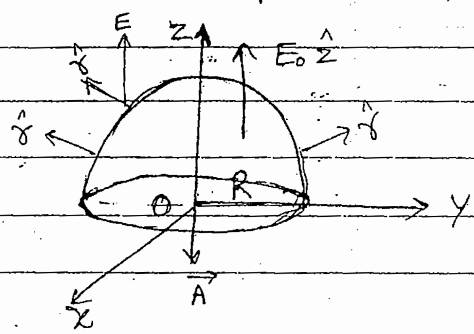
* **Note** :- If Any close surface (sphere, cylender, cube) placed in the uniform electric field then net electric flux passing through the closed surface is zero always.

Net flux = 0

Prob 1 - e.g. - closed hemisphere of radius R is placed in uniform electric field.

Net flux = 0

$\phi_E = 0$



Note :- If we cut a hollow sphere then we'll get a open hemisphere & if we place a disc on it then it become close hemisphere.

Proof :- Area of cross-section $= \pi R^2$

Flux entering in cross-section $= -\pi R^2 E$

$\left\{ \pi R^2 E \cos 180° = -\pi R^2 E \right.$

Let angle b/w $\vec{E_0}$ & $\hat{r}$ is θ.

bcz area vector is changing the dirn.

flux through curved surface

$$\phi_E = \int \vec{E} \cdot d\vec{S}$$

$$= \int E \cdot R^2 \sin\theta \, d\theta \, d\phi$$

$$= ER^2 \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \pi R^2 E$$

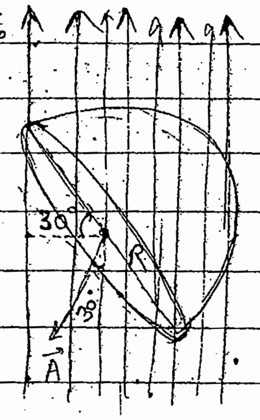
Entering flux :-

Hence Net flux $= -\pi R^2 E + \pi R^2 E = 0$

Note :- Total flux passing through the (closed surface) of hemispherical surface $= 0$

& Electric flux passing through the hemispherical surface $= \pi R^2 E$

Problem :- Calculate the total flux passing through the hemisphere →



Net $\phi_E = 0$

Electric flux passing through the hemispherical surface.

$$\phi_E = \int \vec{E} \cdot d\vec{s}$$

$\left\{ \begin{array}{l} ds = r\,d\theta\,r\,d\phi \\ \int ds = \pi R^2 \end{array} \right.$

$$= E\, S \cos\theta$$

$$= E \pi R^2 \cos 30° = \frac{\sqrt{3}}{2} \pi R^2 E$$

always take angle from horizontal.

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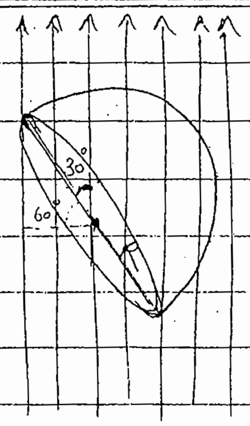
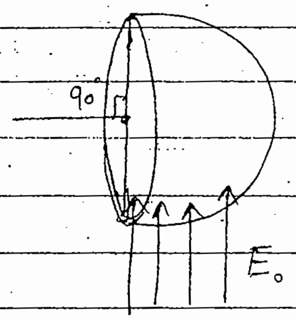
Electric flux passing through the hemispherical surface

$$\phi_E = \pi R^2 E \cos 60^\circ$$
$$= \frac{\pi R^2 E}{2}$$



If hemisphere is tilted at 90°.
then flux

$$\phi_E = 0.$$



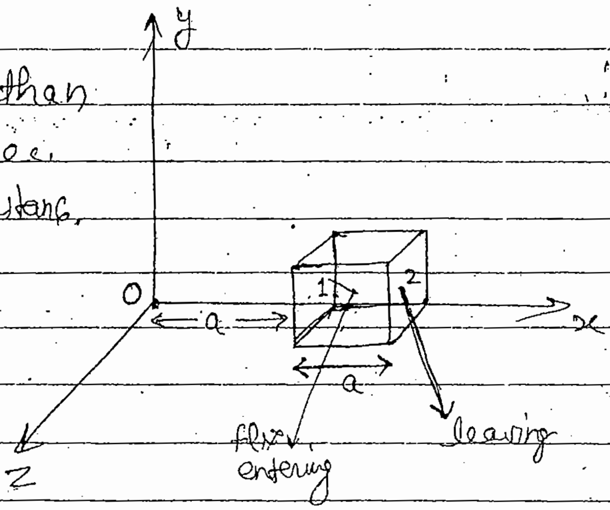
* In Non uniform electric field, Electric flux passing through a closed surface is not zero.

Prob 1- Calculate the electric flux for a cube of side a. as shown in the figure where

$$E_x = b x^{1/2}, \quad E_y = E_z = 0$$
$$a = 10 \text{ cm}$$
$$b = 800 \text{ C/m}^2$$

leaving flux is more than entering flux bcz Elec. field is incresing with distance.

$$\phi = EA\cos\theta$$
$$\phi_{net} = \phi_2 - \phi_1$$
$$= b(2a)^{1/2} a^2 - b(a)^{1/2} a^2$$
$$(A = a^2) \qquad = b a^{5/2} [\sqrt{2} - 1]$$



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$$\phi_{net} = (800)(10)^{5/2}[\sqrt{2}-1]$$

Here, electric flux passing throug the cube is Non-zero bez electric field was non-uniform. bcz in case of uniform electric field, flux is zero.

Net charge enclosed by the cube

$$\phi = \frac{q}{\epsilon_0}$$

$$q = \epsilon_0 b a^{5/2}(\sqrt{2}-1)$$

$$q = +ve$$

becz flux leaving is more.

→ If $E_x = \frac{b}{x^{1/2}}$

then flux entering is more than leaving

$$\phi_{net} = \phi_1 - \phi_2 = \frac{b}{a^{1/2}} \times a^2 - \frac{b}{(2a)^{1/2}} a^2$$

$$\cancel{\frac{b a}{(2a)^{1/2}} a^2} = b a^{3/2}\left[1 - \frac{1}{\sqrt{2}}\right]$$

By using this method we can calculate the charge contained in the surface.

$$q = -\epsilon_0 b a^{3/2}\left[1 - \frac{1}{\sqrt{2}}\right]$$ {flux entering is more so q is -ve}

Prob:- Electric field in some region is given by $\vec{E} = K r^3 \hat{r}$ where K is constant. find

(i) charge density ρ

(ii) Total charge contained in the sphere of radius R.

(i) $\vec{E} = K r^3 \hat{r}$

Diff form of Gauss law $\nabla \cdot E = \frac{\rho}{\epsilon_0}$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) = \frac{\rho}{\epsilon_0}$$

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$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 k r^3) = \frac{\rho}{\epsilon_0}$$

$$\rho = \epsilon_0 \frac{1}{r^2} 5 k r^4$$

$$\boxed{\rho = 5\epsilon_0 k r^2}$$

(ii) Total charge Q

$$Q = \int_V \rho \, d\tau$$

$$= \int_V \rho \, r^2 \sin\theta \, d\theta \, d\phi \, dr$$

ρ is not depending upon θ & ϕ so

$$\int d\tau = 4\pi r^2 dr$$

$$Q = \int_V \rho \, 4\pi r^2 dr = \int_0^R 5\epsilon_0 k r^2 \, 4\pi r^2 dr$$

$$Q = 5\epsilon k \times 4\pi \times \frac{R^5}{5}$$

$$\boxed{Q = 4\pi\epsilon_0 k R^5}$$

Prob 1- Electric field in some region is given by

$$\vec{E} = \frac{(A\hat{r} + B\sin\theta\cos\phi\,\hat{\phi})}{r}$$

Where A & B are constants, find
(i) ρ
(ii) Total charge contained in the sphere of radius R.

(i) $$\vec{E} = \frac{A}{r}\hat{r} + \frac{B}{r}\sin\theta\cos\phi\,\hat{\phi}$$

$$\nabla\cdot E = \frac{\rho}{\epsilon_0}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}E_\phi = \frac{\rho}{\epsilon_0}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \cdot \frac{A}{r}\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{B}{r}\sin\theta\cos\phi\right] = \frac{\rho}{\epsilon_0}$$

$$\frac{1}{r^2} A + \frac{1}{r\sin\theta}\frac{B}{r}\sin\theta(-\sin\phi) = \frac{\rho}{\epsilon_0}$$

$$\frac{A}{r^2} + \frac{B}{r^2}(-\sin\phi) = \frac{\rho}{\epsilon_0}$$

$$\boxed{\rho = \frac{\epsilon_0}{r^2}\left[A - B\sin\phi\right]}$$

(ii) $$Q = \int_V \rho\, d\tau$$

$$= \int_V \frac{\epsilon_0}{r^2}(A - B\sin\phi)\, r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$$= \epsilon_0 \int_0^R\int_0^\pi\int_0^{2\pi} (A - B\sin\phi)\sin\theta\, d\theta\, d\phi\, dr$$

$$= \epsilon_0 R \int_0^\pi\int_0^{2\pi} \left(A\sin\theta\, d\theta\, d\phi - B\sin\phi\, \sin\theta\, d\theta\, d\phi\right)$$

$$= \epsilon_0 R\left[A\left[-\cos\theta\right]_0^\pi [2\pi] - B\left[-\cos\phi\right]_0^{2\pi}\left[-\cos\theta\right]_0^{\pi}\right]$$

$$= \epsilon_0 R\left[-2\pi A(-1-1) - B(1-1)(-1-1)\right]$$

$$= \epsilon_0 R(4\pi A)$$

$$= 4\pi\epsilon_0 A R$$

Prob :- Electric field in some region is given by

$$\vec{E} = \frac{A\hat{r}}{r^2} + \frac{B\sin\theta\cos\phi}{r}\hat{\phi}$$

Find (i) ρ (ii) Q

(i) $$\nabla\cdot E = \frac{\rho}{\epsilon_0}$$

$$\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{A}{r^2}\right)}_{\nabla\cdot\left(A\frac{\hat{r}}{r^2}\right)} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{B}{r}\sin\theta\cos\phi\right) = \frac{\rho}{\epsilon_0}$$

$$\underbrace{\frac{1}{r^2}(0)}_{A\nabla\cdot\left(\frac{\hat{r}}{r^2}\right)} + \frac{1}{r\sin\theta}\cdot\frac{B}{r}\sin\theta(-\sin\phi) = \frac{\rho}{\epsilon_0}$$

$$A\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) - \frac{B}{r^2}\sin\phi = \frac{\rho}{\epsilon_0}$$

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$$\rho = \epsilon_0\left[A\,\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) - \frac{B}{r^2}\sin\phi\right]$$

$$\rho = \epsilon_0\left[A\cdot 4\pi\delta^3(r) - \underbrace{\frac{B\sin\phi}{r^2}}_{Q=0}\right]$$

(ii) $Q = \int_V \rho\, d\tau$ when ~~limit enclosing the~~ surface $\int_V \delta^3(r)\,d\tau = 1$

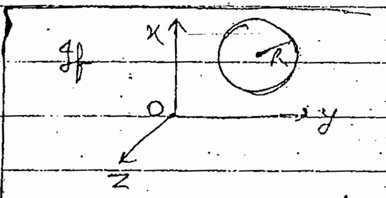
$$Q = \epsilon_0 4\pi A\int_0^R \delta^3(r)\,d\tau$$

$r \to 0$ to R

$$= \epsilon_0 4\pi A\int_0^R \delta^3(r)\, 4\pi r^2\,dr$$

$$\boxed{Q = \epsilon_0 4\pi A}$$

If



$\int_V \delta^3(r)\,d\tau = 0$ bcz limit is not enclosing the sphere

Ques:- $\vec{E} = \dfrac{Ae^{-4r}}{r^2}\hat{r} + B\dfrac{\sin\theta\cos\phi}{r}$

$$\rho = \epsilon_0\left[\nabla\cdot\left(\frac{Ae^{-4r}}{r^2}\hat{r}\right) + \nabla\cdot\left(\frac{B\sin\theta\cos\phi}{r}\right)\right]$$

$$\nabla\cdot\left[A\,\overset{\text{II}}{e^{-4r}}\left(\overset{\text{I}}{\frac{\hat{r}}{r^2}}\right)\right] = A\left[\frac{\hat{r}}{r^2}\cdot\nabla(-4e^{-4r}) + e^{-4r}(4\pi\delta^3(r))\right]$$

$$\vec{\nabla}\cdot(\lambda\vec{A}) = \vec{A}\cdot\vec{\nabla}\lambda + \lambda\vec{\nabla}\cdot\vec{A}$$

$$= -\frac{4A}{r^2}e^{-4r} + A4\pi\delta^3(r)e^{-4r}$$

$$= -\frac{4A}{r^2}e^{-4r} + A4\pi\delta^3(r) \quad \{f(x)\delta(x) = f(0)\,\delta(x)$$

$$\rho = \epsilon_0\left[-\frac{4A}{r^2}e^{-4r} + A4\pi\delta^3(r) - \frac{B\sin\phi}{r^2}\right]$$

$$Q = \int_V \rho\,d\tau = -4A\epsilon_0\int_0^R \frac{e^{-4r}}{r^2}4\pi r^2\,dr + 4\pi A\epsilon_0\int_0^R \delta^3(r)$$

$$= -16\pi A\epsilon_0\left[\frac{e^{-4r}}{-4}\right]_0^R + 4\pi\epsilon_0 A$$

$$= 4\pi\epsilon_0 A\left[e^{-4R} - 1\right] + 4\pi\epsilon_0 A$$

$$= 4\pi A\epsilon_0 e^{-4R}$$

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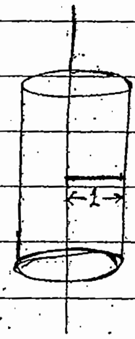
Prob 1:- Calculate the charge density at $(1, \frac{\pi}{4}, 3)$ & the total charge enclosed by the cylender of radius 1 meter with $-2 \le z \le 2m$.

given that, electric field

$$\vec{E} = s z \cos^2\phi \, \hat{z} \; N/C$$



$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

$$\frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0}$$

$$\frac{\partial}{\partial z}(s z \cos^2\phi) = \frac{\rho}{\epsilon_0}$$

$$s\cos^2\phi = \frac{\rho}{\epsilon_0}$$

$$\rho = \epsilon_0 \, s\cos^2\phi$$

$$\rho = \epsilon_0 (1)\left(\cos\frac{\pi}{4}\right)^2$$

$$\rho = \frac{\epsilon_0}{2} \quad c/m^3 \quad \underline{\underline{Ans}}$$

$$\Phi = \int_V \rho \, d\tau = \int_0^1\int_0^{2\pi}\int_{-2}^{2} \rho \, s \, ds \, d\phi \, dz$$

$$= \int_0^1\int_0^{2\pi}\int_{-2}^{2} \epsilon_0 \, s\cos^2\phi \, s \, ds \, d\phi \, dz$$

$$= \epsilon_0 \left[\frac{s^3}{3}\right]_0^1 \int_0^{2\pi} \left(\frac{1+\cos 2\phi}{2}\right) d\phi \int_{-2}^{2} dz$$

$$= \epsilon_0 \frac{1}{3} \frac{1}{2} \left\{ [\phi]_0^{2\pi} + \left[\frac{\sin 2\phi}{2}\right]_0^{2\pi} \right\} [z]_{-2}^{2}$$

$$= \epsilon_0 \frac{1}{3} \frac{1}{2} [2\pi + 0] (2+2)$$

$$= \frac{4}{3}\pi\epsilon_0 \quad c \quad \underline{\underline{Ans}}$$

$\int^{R} \delta(r) d\tau$

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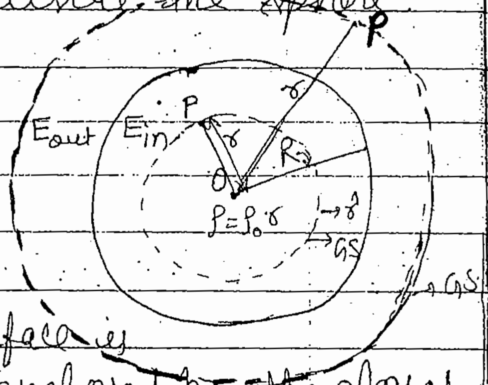
Note: In Problems based on symmetry, we can easily find out electric field using Gauss law but if symmetry is not present then we use Coulomb's law to calculate electric field.

Que ①: If sphere of radius R carries a vol. charge density $\rho = \rho_0 r$ where ρ is the constant & r is the distance from the centre of the charged sphere. Calculate the electric field both inside & outside the sphere.

Soln:-



There is spherical symmetry.

$$\oint \vec{E} \cdot d\vec{S} = \frac{q_{enc}}{\epsilon_0}$$

The flux passing through the surface is depend upon the total charge enclosed by the closed surface.

inside (i)

charge enclosed by the sphere of radius r $q_{enc} = ?$

$$q_{enc} = \oint = \int_V \rho \, d\tau = \int_0^r \int_0^{2\pi} \int_0^{\pi} \rho_0 r \, r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \int_0^r \rho_0 r \, 4\pi r^2 dr = \int_0^r \rho_0 4\pi r^3 dr$$

$$= \rho_0 4\pi \left[\frac{r^4}{4}\right]_0^r = \rho_0 4\pi \frac{r^4}{4}$$

$$= \pi \rho_0 r^4$$

$$\oint \vec{E} \cdot d\vec{S} = \frac{q_{enc}}{\epsilon_0}$$

$$\vec{E} \cdot 4\pi r^2 = \frac{\pi \rho_0 r^4}{\epsilon_0}$$

$$\boxed{\vec{E}_{in} = \frac{\rho_0 r^2}{4 \epsilon_0} \hat{r}}$$

(ii) Outside the sphere

$ds \rightarrow 4\pi r^2$ This r is the distance from O to P.

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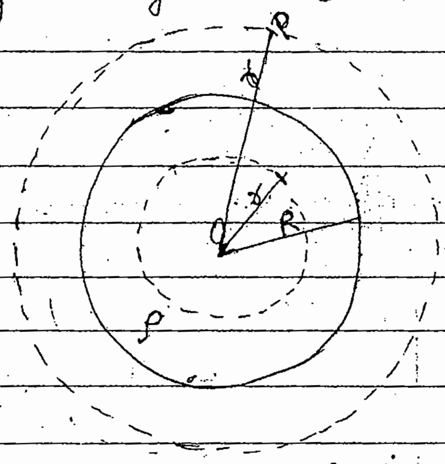
$$q_{enc} = \int_V \rho \, d\tau = \int_0^R \rho_0 r \, 4\pi r^2 \, dr$$
$$= \rho_0 4\pi \left[\frac{r^4}{4}\right]_0^R = \rho_0 \pi R^4$$

$$\oint \vec{E} \cdot d\vec{s} = \frac{q_{enc}}{\epsilon_0}$$
$$E \cdot 4\pi r^2 = \frac{\pi \rho_0 R^4}{\epsilon_0}$$
$$\boxed{\vec{E}_{out} = \frac{\rho_0 R^4}{4\epsilon_0 r^2}\hat{r}}$$

Hence $E_{in} \propto r^2$ & $E_{out} \propto \frac{1}{r^2}$

* If there is spherical symmetre & ρ is given r^n ~~then~~ ($\rho \propto r^n$) then electric field inside will be
$$E_{in} \propto r^{n+1}$$
& outside will be $E_{out} \propto \frac{1}{r^2}$

Ques ② Find the electric field inside & outside of a uniformly charged sphere of radius R and total charge q.
[uniformly charged → ρ is constant with R]



We have to develope a relation b/w ρ & q

$$q = \int_V \rho \, d\tau = \int_0^R \rho \, \cancel{r} \, 4\pi r^2 \, dr = 4\pi \rho \left(\frac{r^3}{3}\right)_0^R$$

$= \int \times \frac{1}{3}\pi R$

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$$q = \frac{4}{3}\pi R^3 \rho$$

ρ is uniform in the sphere of radius R so we can directly write $q = \frac{4}{3}\pi R^3 \rho \Rightarrow \rho = \frac{q}{\frac{4}{3}\pi R^3}$

(i) Electric field inside the sphere

~~q_{enc} in the sphere of radius r will be~~

$$\cancel{q_{enc} = \frac{4}{3}\pi r^3 \rho}$$

$$q_{enc} = \int_V \rho \, d\tau = \int_0^r \frac{q}{\frac{4}{3}\pi R^3} 4\pi r^2 dr$$

$$= \frac{3q}{R^3}\left[\frac{r^3}{3}\right]_0^r = \frac{q r^3}{R^3}$$

$$q_{enc} = \frac{q r^3}{R^3}$$

~~when r = R then total charge enclosed within this sphere is~~

$$\oint \vec{E} \cdot d\vec{s} = \frac{q_{enc}}{\epsilon_0}$$

$$E \cdot 4\pi r^2 = \frac{1}{\epsilon_0}\frac{q r^3}{R^3}$$

$$E = \frac{q r}{4\pi R^3 \epsilon_0}$$

$$\boxed{\vec{E}_{in} = \frac{q r}{4\pi \epsilon_0 R^3}\hat{r}}$$

(ii) Electric field outside the sphere,

$$q_{enc} = \int_V \rho \, d\tau = \int_0^R \frac{q}{\frac{4}{3}\pi R^3} 4\pi r^2 dr$$

$$= \frac{3q}{R^3}\left[\frac{r^3}{3}\right]_0^R = \frac{q}{R^3} R^3$$

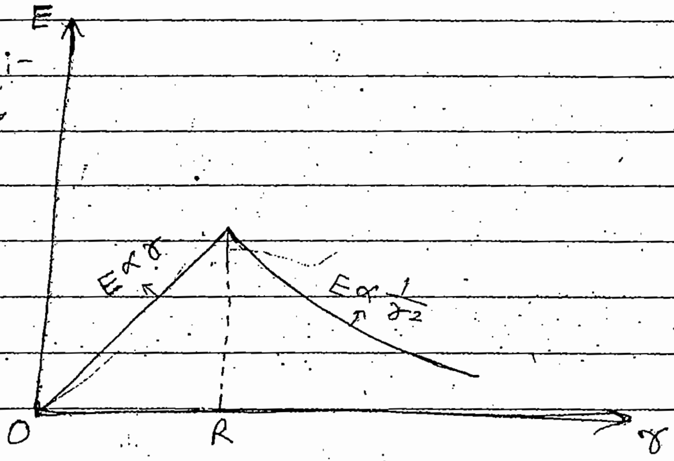
$$q_{enc} = q$$

$$\oint \vec{E} \cdot \vec{ds} = \frac{q_{enc}}{\epsilon_0}$$

$$E \cdot 4\pi r^2 = \frac{q}{\epsilon_0}$$

$$\boxed{\vec{E}_{out} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}}$$

Plot of Electric field :-



Q.(2) :- find the electric field for spherical charge distribution given by

$$\rho(r) = \begin{cases} \rho_0\left(1 - \frac{r^2}{a^2}\right) & r \leq a \\ 0 & r > a \end{cases}$$

where r is the distance from the centre of the sphere & a is the radius of the sphere. Also find the value of r for which electric field will be maximum.

(i) Inside the sphere,

$$q_{enc} = \int_V \rho \, d\tau = \int_0^r \rho_0\left(1 - \frac{r^2}{a^2}\right) 4\pi r^2 dr$$

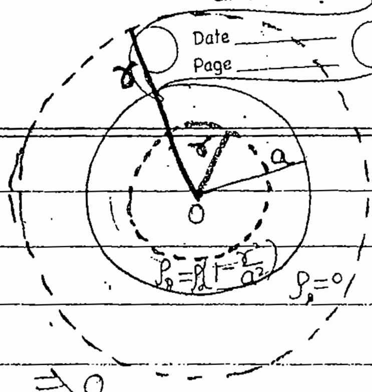
$$= \rho_0 4\pi \left[\frac{r^3}{3} - \frac{1}{a^2}\frac{r^5}{5}\right]_0^r$$

$$= \rho_0 4\pi \left[\frac{r^3}{3} - \frac{r^5}{5a^2}\right]$$

$$\oint \vec{E} \cdot \vec{ds} = \frac{q_{enc}}{\epsilon_0}$$

$$E \cdot \cancel{4\pi} r^2 = \frac{1}{\epsilon_0} \rho_0 \cancel{4\pi} \left[\frac{r^3}{3} - \frac{r^5}{5a^2}\right]$$

$$\boxed{\vec{E}_{in} = \frac{\rho_0}{\epsilon_0}\left[\frac{r}{3} - \frac{r^3}{5a^2}\right]\hat{r}}$$



(ii) Outside the sphere,

$$q_{enc} = \int \cancel{\rho\, d\tau} = \int_0^{a} \cancel{\rho\, 4\pi r^2 dr} = 0$$

$$\cancel{E \neq 0}$$

$$q_{enc} = \int_0^a \rho_0\left(1-\frac{r^2}{a^2}\right)4\pi r^2 dr$$

$$= \rho_0 4\pi\left[\frac{a^3}{3} - \frac{a^3}{5}\right]$$

$$= 4\pi\rho_0 a^3 \times \frac{2}{15} = \frac{8}{15}\pi\rho_0 a^3$$

$$E\cdot\cancel{4\pi} r^2 = \frac{\overset{2}{\cancel{8}}}{15}\cancel{\pi}\rho_0 a^3\frac{1}{\epsilon_0}$$

$$\boxed{\vec{E}_{out} = \frac{2\rho_0 a^3}{15\epsilon_0 r^2}\hat{r}}$$

The value of r at which field is maximum,

$$\frac{\partial E_{in}}{\partial r} = 0$$

$$\frac{\rho_0}{\epsilon_0}\left[\frac{1}{3} - \frac{1}{5a^2}3r^2\right] = 0$$

$$\frac{1}{5a^2}3r^2 = \frac{1}{3} \Rightarrow r^2 = \frac{5}{9}a^2$$

$$\boxed{r = \frac{\sqrt{5}}{3}a}$$

* **Note :-**

electric field can not be maximum, outside the sphere because outside electric field is decreasing as $E_{out} \propto \frac{1}{r^2}$.

Note- We can use Gauss law to find elec. field only if cylender is long

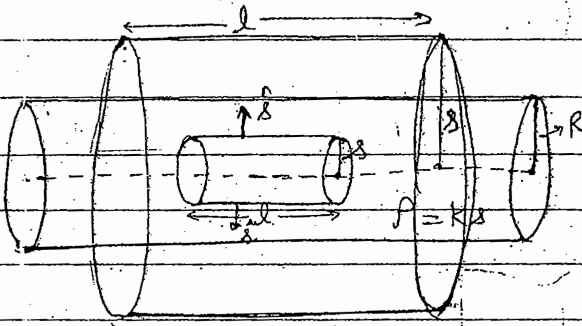
CYLENDER

Q.1:- A long cylender of rad. R carries a charge density i.e. proportional to the distance from the axis

$\rho = k s$

where k is constant.

Find the electric field inside & outside the cylender



(i) Total charge enclosed by the Gaussian surface:

$$q_{enc} = \int_V \rho \, d\tau$$

$$= \int_0^s \int_0^{2\pi} \int_0^l k s \; s \, ds \, d\phi \, dz$$

$$= k \left(\frac{s^3}{3}\right)_0^s (\phi)_0^{2\pi} (z)_0^l = \frac{k}{3} s^3 2\pi \cdot l$$

$$q_{enc} = \frac{2\pi k s^3 l}{3}$$

Electric field $\oint \vec{E} \cdot d\vec{s} = \frac{q_{enc}}{\epsilon_0}$

flux only pass through the curve surface of cylender bcz dir^n of $\vec{E}$ is always normal to the axis. Along Curved surface $\vec{E}$ is normal to axis.

So $\cancel{E \cdot 2\pi s l} \; \oint E \cdot s \, d\phi \, dz = \frac{q_{enc}}{\epsilon_0}$

$$E \cdot 2\pi s l = \frac{2\pi k s^3 l}{3\epsilon_0}$$

$$\boxed{\vec{E}_{in} = \frac{k s^2}{3\epsilon_0} \cdot \hat{s}}$$

(ii) Outside the cylender,

$$q_{enc} = \int_V \rho \, d\tau = \int_0^R \int_0^{2\pi} \int_0^l k s \, s \, ds \, d\phi \, dz$$

$$= \frac{2\pi k R^3 l}{3}$$

$$\oint E \cdot ds = \frac{q_{enc}}{\epsilon_0}$$

$$E \cdot 2\pi s l = \frac{1}{\epsilon_0} \frac{2\pi k R^3 l}{3}$$

$$\boxed{\vec{E}_{out} = \frac{k R^3}{3 \epsilon_0 s} \hat{s}}$$

Note :- If $\rho \propto s^n$

then $\boxed{E_{in} \propto s^{n+1} \quad E_{out} \propto \frac{1}{s}}$

e.g. If given $\rho \propto s^3$

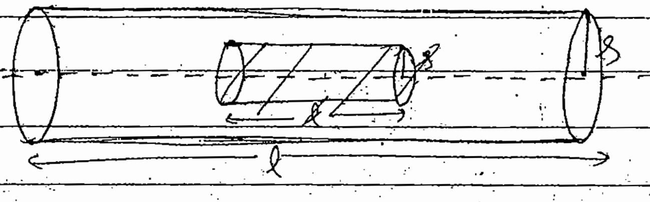
then $n = 3$

but $\rho = \rho_0 \left(1 - \frac{s^3}{a^2}\right)$ then there is no clear dependence of r. We can' not use this method.

*. To find out $\vec{E}$ outside the cylender Gaussian surface should be infinitely long including cross-section also for Gauss law. (complicated).

We use a short cylender which include curved surface only.

Qu: 2 :- Find the $\vec{E}$ at a distance s from an infinitely long straight wire carrying a uniform line charge density λ.



$q_{enc} = \int \lambda \, dl$ (suppose length → l)

$= \lambda l$

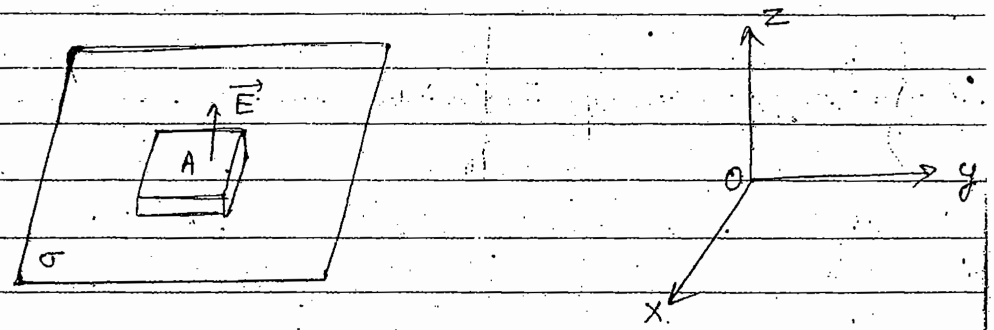
$\oint \vec{E} \cdot d\vec{s} = \frac{q_{enc}}{\epsilon_0}$

$E \cdot 2\pi s l = \frac{\lambda l}{\epsilon_0}$

$$\boxed{\vec{E} = \frac{\lambda}{2\pi \epsilon_0 s} \hat{s}}$$

Plane Surface

Q.1 :- An infinite plane carries a uniform surface charge density σ. Find its electric field.



dir^n of $\vec{E}$ can be $+\hat{z}$ or $-\hat{z}$

(Gaussian surface → Gaussian pill box) half part is upper & half part is lower to the plane.

$\vec{E}$ will pass from area A only. so flux will pass through area A.

charge enclosed $q_{enc} = \sigma A$

$$\oint E \cdot dS = \frac{q}{\epsilon_0}$$

flux = 2 EA from −z to z.

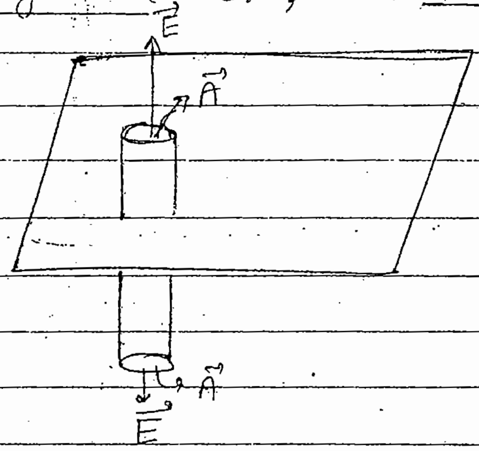
$$[EA + EA] = \frac{\sigma A}{\epsilon_0}$$

$$2EA = \frac{\sigma A}{\epsilon_0}$$

$$\boxed{\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{n}}$$

$\hat{n} \rightarrow$ dir^n of $\vec{E}$ normal to the surface.

* In place of Gaussian pill box we can also use a Gaussian cylender also, as a Gaussian surface.



Gaussian surface must be symmetrical from the point so it should be half up & half down.

Que :- A hollow spherical shell carries a charge density $\rho = \frac{K}{r^2}$, $a \leq r \leq b$. Find the electric field in 3 regions :-

(i) $r < a$

(ii) $a < r < b$

(iii) $r > b$

Thick hollow sphere → have 2 radius
Thin " " → have 1 radius (inner & outer radius same)

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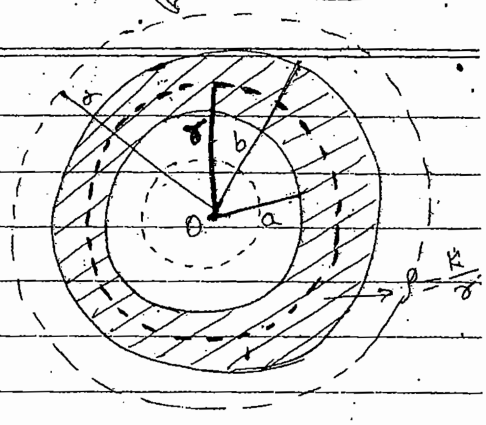


(i) $q_{enc} = \int_V \rho \, d\tau$

$= \int_0^{r} k(0)\, 4\pi r^2 dr$

$= \cancel{4\pi k a}\; 0$

$\oint E \cdot ds = \frac{q_{enc}}{\epsilon_0} \Rightarrow \boxed{E = 0}$

(ii) $q_{enc} = \int_V \rho \, d\tau = \int_a^r \frac{k}{r^2} 4\pi r^2 dr$

$= \int_0^a k\, 4\pi dr + \int_a^r k \cdot 4\pi dr$
$\Downarrow$
0

$= 4\pi k (r - a)$

$\oint E \cdot ds = \frac{q_{enc}}{\epsilon}$

$E \cdot 4\pi r^2 = \frac{4\pi k (r-a)}{\epsilon_0}$

$$\boxed{\vec{E} = \frac{k}{\epsilon_0 r^2}(r-a)\hat{r}}$$

(iii) $q_{enc} = \int_V \rho \, d\tau = \int_a^b \frac{k}{r^2} 4\pi r^2 dr$

$= 4\pi k (b-a)$

$\oint E \cdot ds = \frac{q_{enc}}{\epsilon_0}$

$E \cdot 4\pi r^2 = \frac{4\pi k (b-a)}{\epsilon_0}$

$$\boxed{\vec{E} = \frac{k}{\epsilon_0 r^2}(b-a)\hat{r}}$$

$\vec{E}$ is Maximum for $\frac{\partial \vec{E}_{(ii)}}{\partial r} = 0$

$$\frac{\partial}{\partial r}\left[\frac{K}{\epsilon_0 r^2}(r-a)\right] = 0$$

$$\frac{K}{\epsilon_0}\left[\frac{-1}{r^2}+\frac{2a}{r^3}\right] = 0$$

$$\frac{K}{\epsilon_0 r^2}\left(-1+\frac{2a}{r}\right) = 0$$

$$\frac{2a}{r} = +1$$

$$\boxed{r = +2a}$$

$$E_{Max} = \frac{K}{\epsilon_0 (2a)^2}(2a-a)$$

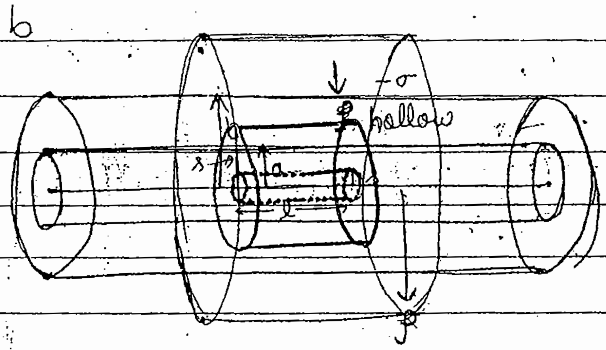
$$\boxed{E_{max} = \frac{K}{4a\epsilon_0}}$$

Ques :- A long co-axial cable carries a uniform volume charge density ρ on inner cylender (radius a) & a uniform surface charge density ($-\sigma$) on the outer cylender (radius b) surface charge is such that cable as a whole is electrically neutral. Find the electric field in 3 regions-

(i) $s < a$

(ii) $a < s < b$

(iii) $s > b$



(I) $s < a$

$$q_{enc} = \int_V \rho \, d\tau = \int_0^s\int_0^{2\pi}\int_0^l \rho \, s\,ds\,d\phi\,dz$$

$$= \rho\,\frac{s^2}{2}\,(z)_0^l\,2\pi$$

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$$q_{enc} = \rho \pi s^2 l$$

$$\oint \vec{E} \cdot d\vec{s} = \frac{q_{enc}}{\epsilon_0}$$

$$E \cdot 2\pi s l = \frac{\rho \pi s^2 l}{\epsilon_0}$$

$$\boxed{\vec{E} = \frac{\rho s}{2\epsilon_0}\hat{s}}$$

(ii) $a < s < b$

$$q_{enc} = \int_0^a \int_0^{2\pi} \int_0^l \rho\, s\, d\phi\, dz\, ds$$

$$= \rho \pi a^2 l$$

$$E \cdot 2\pi s l = \frac{\rho \pi a^2 l}{\epsilon_0}$$

$$\boxed{\vec{E} = \frac{\rho a^2}{2\epsilon_0 s}\hat{s}}$$

(iii) $s > b$

$$q_{enc} = 0$$

So $\boxed{E = 0}$

Que 1 ① The electric field due to an unknown charge distribution is given by $\vec{E} = \frac{q}{r^2} e^{-4r}\hat{r}$.

Find charge density ρ & total charge in the sphere of radius R centred at the origin.

Que 1 ② A charged ball of radius R carries a charge density $\rho = 2\epsilon_0\left(3a + \frac{b}{r}\right)$

(i) Total charge in the ball.

(ii) Electric field inside & outside the ball

(iii) Net no. of electric field lines passing through the ball.

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① $\vec{E} = \frac{q}{r^2} e^{-4r}\,\hat{r}$

ρ & Q = ?

$\nabla\cdot E = \frac{\rho}{\epsilon_0}$

$\nabla\cdot\left(\frac{q}{r^2}e^{-4r}\hat{r}\right) = \frac{\rho}{\epsilon_0}$

$q\left[\frac{\hat{r}}{r^2}\nabla\cdot(e^{-4r}) + e^{-4r}\nabla\cdot\left(\frac{\hat{r}}{r^2}\right)\right] = \frac{\rho}{\epsilon_0}$

$q\left[\frac{\hat{r}}{r^2}(-4e^{-4r}) + e^{-4r}(4\pi\,\delta^3(r))\right] = \frac{\rho}{\epsilon_0}$

$-\frac{4q}{r^2}e^{-4r} + q\,4\pi\,\delta^3(r)\,e^{-4r} = \frac{\rho}{\epsilon_0}$

$\boxed{\rho = \epsilon_0 q\left[4\pi\,\delta^3(r) - \frac{4}{r^2}e^{-4r}\right]}$ $\{f(x)\delta(x) = f(0)\delta(x)\}$

Now $Q = \int \rho\, d\tau = -4q\epsilon_0\int_0^R \frac{e^{-4r}}{r^2}4\pi r^2\, dr + 4\pi q\epsilon_0\int_0^R \delta^3(r)\,dr$

$= -16q\epsilon_0\pi\left[\frac{e^{-4r}}{-4}\right]_0^R + 4\pi q\epsilon_0$

$= +4q\epsilon_0\pi\,[e^{-4R}-1] + 4\pi q\epsilon_0$

$= 4\pi\epsilon_0 q\,[e^{-4R}-1+1]$

$\boxed{Q = 4\pi\epsilon_0 q\, e^{-4R}}$

② $\rho = 2\epsilon_0\left(3a + \frac{b}{r}\right)$

(i) Total charge in the ball

$q = \int_V \rho\, d\tau \Rightarrow q = \int_0^R 2\epsilon_0\left(3a+\frac{b}{r}\right)4\pi r^2\, dr$

$q = 4\pi\, 2\epsilon_0\left[3a\frac{r^3}{3} + b\frac{r^2}{2}\right]_0^R$

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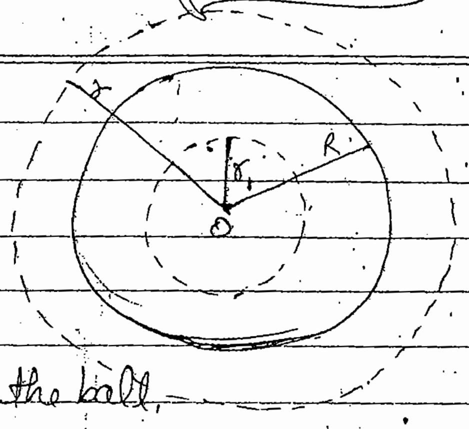
$$q = 8\pi\epsilon_0\left[aR^3 + \frac{b}{2}R^2\right]$$

$$\boxed{q = 8\pi\epsilon_0 R^2\left[aR + \frac{b}{2}\right]}$$



(ii) Electric field inside 4 outside the ball.

Inside :-

$$q = \int_V \rho\, d\tau = \int_0^r 2\epsilon_0\left(3a + \frac{b}{r}\right)4\pi r^2 dr$$

$$= 8\pi\epsilon_0\left[ar^3 + \frac{b}{2}r^2\right]$$

$$\oint E\cdot ds = \frac{q}{\epsilon_0}$$

$$E\cdot 4\pi r^2 = 8\pi\left[ar^3 + \frac{b}{2}r^2\right]$$

$$\boxed{\vec{E} = 2\left(ar + \frac{b}{2}\right)\hat{r}}$$

Outside :-

$$q = 8\pi\epsilon_0 R^2\left[aR + \frac{b}{2}\right]$$

$$\oint E\cdot ds = \frac{q}{\epsilon_0}$$

$$E\cdot 4\pi r^2 = 8\pi R^2\left[aR + \frac{b}{2}\right]$$

$$\boxed{E = \frac{2R^2}{r^2}\left(aR + \frac{b}{2}\right)\hat{r}}$$

(iii) Net no. of electric field lines passing through the ball.

Net no. of elec. field lines = flux

$$\text{flux} = \oint E\cdot ds = \frac{q}{\epsilon_0}$$

$$= \frac{8\pi\epsilon_0 R^2\left[aR + \frac{b}{2}\right]}{\epsilon_0}$$

$$= 8\pi R^2\left[aR + \frac{b}{2}\right]$$

Neutral Points :- Neutral points are those points at which electric field is zero.

$\vec{E} = 0$

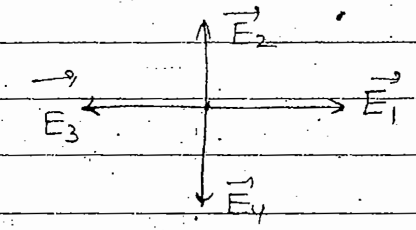
If we place a charge at the neutral point, force on the charge will be zero

$\vec{F} = q\vec{E}$

$F = 0$

By Superposition principle :-
Coulomb force (or electric field) follow the superposition principle.

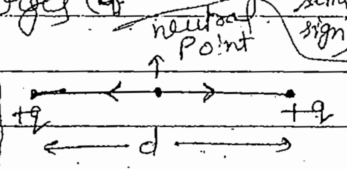
If



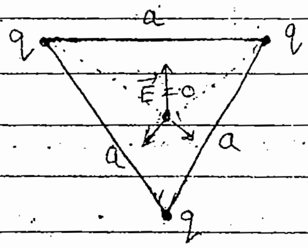
then electric field can be vectorially sum

$\vec{E} = \vec{E_1} + \vec{E_2} + \vec{E_3} + \vec{E_4} + \cdots$

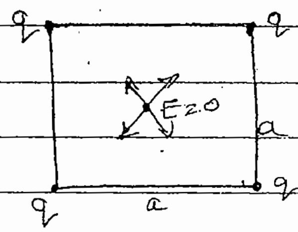
Suppose we have two point charges (of same mag. & same sign)
Electric field due to L.H.S. charge →
R.H.S. " ←
So, Electric field is zero



Similarly for 3 charges,
At the centre $\vec{E} = 0$, This is based on symmetry.



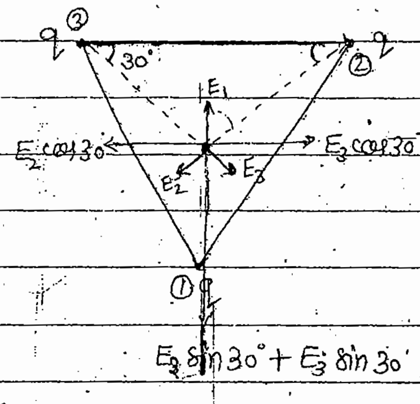
If there are 4 charges
At centre $\vec{E} = 0$
Similarly, If we have, regular pentagon then at centre $\vec{E} = 0$



*** Digonal of a square of sides a will be a√2.

1 Triangle is the polygon of 3 sides &
Square " " " 4 "

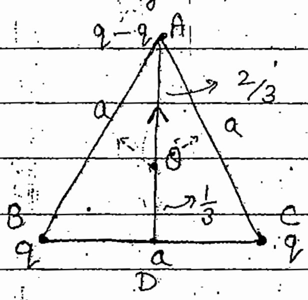
* If we have any regular polygon then at its centre $\vec{E} = 0$.



→ The side angle are of 60°.
→ Magnitude of $E_1 = E_2 = E_3$ are same.
2 → The resultant field at the centre is 0.

$$\vec{E} = 0$$

* Suppose we have 2 charges at B & C, & No charge at A.

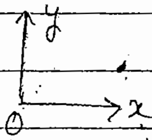


$$AD = \frac{\sqrt{3}}{2}a$$

$$AO = \frac{2}{3} \times \frac{\sqrt{3}}{2}a$$

$$AO = \frac{a}{\sqrt{3}}$$

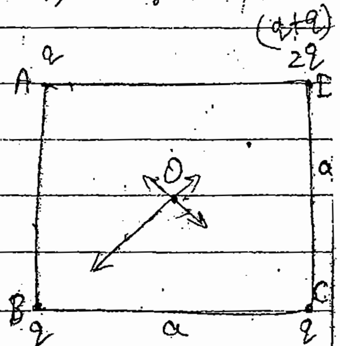
At A, No charge i.e. $+q-q$
The dir^n of $\vec{E}$ will be perpendicular to the charge at B & C so
dir^n of $\vec{E} \rightarrow \hat{y}$



$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{3q}{a^2}\hat{y}}$$

A → +q, −q
+q charge at A, B, C makes $\vec{E} = 0$ at O.
Now $\vec{E}$ will be only due to (−q) charge at A.

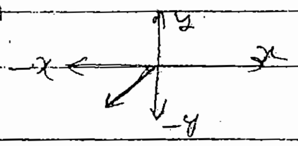
* $2q = q + q$



$\vec{E}$ due to +q at D,

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2q}{a^2}\left(\frac{-(\hat{x}+\hat{y})}{\sqrt{2}}\right)$$

Dir^n



(***) $DO = \frac{a}{\sqrt{2}}$

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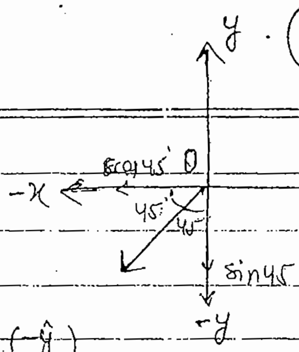


Magnitude of this vector

$r = a/\sqrt{2}$

& vector =

$\frac{a}{\sqrt{2}}\cos 45(-\hat{x}) + \frac{a}{\sqrt{2}}\sin 45°(-\hat{y})$

$\frac{a}{\sqrt{2}}\frac{1}{\sqrt{2}}(-\hat{x}) + \frac{a}{\sqrt{2}}\frac{1}{\sqrt{2}}(-\hat{y})$

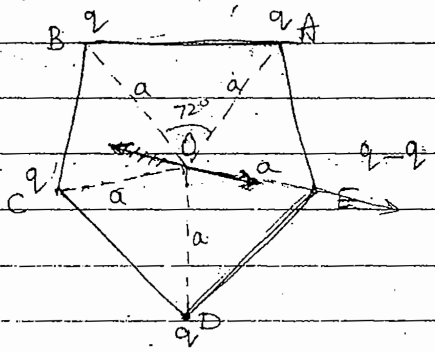
$\vec{r} = -\frac{a}{2}\hat{x} - \frac{a}{2}\hat{y}$

Now $\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$

$= \frac{q}{4\pi\epsilon_0\left(\frac{a^3}{2\sqrt{2}}\right)}\left[-\frac{a}{2}(\hat{x}+\hat{y})\right]$

$$\boxed{\vec{E} = -\frac{q\sqrt{2}}{4\pi\epsilon_0 a^2}\left[-(\hat{x}+\hat{y})\right]}$$

* Find the electric field at O. No charge at E.



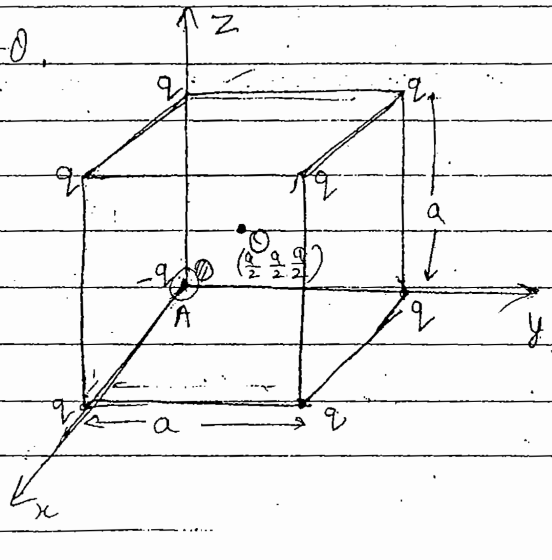
Suppose at pt E, charge is $q-q$.

For $5 \to q$, $\vec{E}=0$ & $\vec{E}$ at O will be due to $-q$ at E pt.

$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{a^2}(\hat{OE})$

Dir^n of $\vec{E} \to (+\hat{OE})$

Prob:- Find the electric field $\vec{E}$ at O.



At A, charge $= -q$

$= -2q+q$

Field will be due to $-2q$.

$A = (0,0,0)$

$O = \left(\frac{a}{2},\frac{a}{2},\frac{a}{2}\right)$

$\vec{r} = \vec{OA} = -\frac{a}{2}\hat{x} - \frac{a}{2}\hat{y} - \frac{a}{2}\hat{z}$

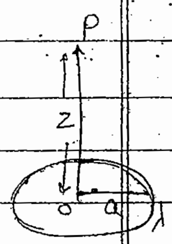
$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\vec{r}$

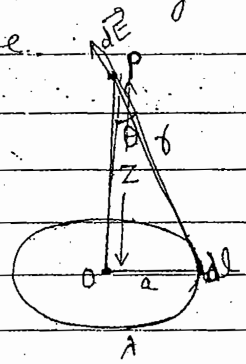
$$|\vec{r}| = \frac{\sqrt{3}\,a}{2}$$

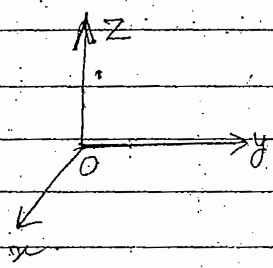
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\,\frac{2q}{\left(\frac{3\sqrt{3}\,a^3}{2^3}\right)}\left[-\frac{a}{2}(\hat{x}+\hat{y}+\hat{z})\right]$$

$$\boxed{\vec{E} = \frac{8q}{4\pi\epsilon_0(3\sqrt{3}\,a^2)}\left[-(\hat{x}+\hat{y}+\hat{z})\right]}$$

Que 1- Electric field of a circular ring of radius a having uniform line charge density λ at a distance z above the centre.







Circular ring is not a closed surface so can not apply Gauss' law.

Let a small element on the ring dl. charge on dl $\rightarrow$ $dq = \lambda\, dl$

This is a point charge so electric field will be pointed away from the charge.

$$dE = \frac{1}{4\pi\epsilon_0}\,\frac{dq}{r^2}$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{\lambda\, dl}{(a^2+z^2)}$$

Now,

Take a similar ~~angle~~ element on opposite side.

Sine comp. will cancle out

$$\cos\theta = \frac{z}{r} = \frac{z}{(a^2+z^2)^{1/2}}$$

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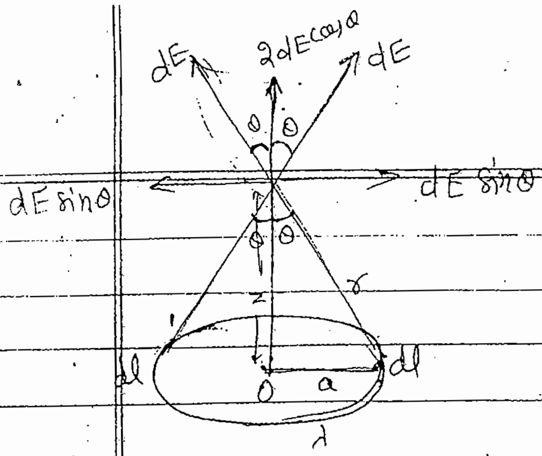

Total Electric field,

$$E = \int 2\, dE \cos\theta$$

$$E = \int \frac{1}{4\pi\epsilon_0} \frac{2\lambda\, dl\, z}{(z^2+a^2)^{3/2}}$$

$$E = \frac{2(\pi a)\lambda z}{4\pi\epsilon_0 (z^2+a^2)^{3/2}}$$

$\{ \int dl = \pi a \}$

$$\boxed{E = \frac{\lambda a z}{2\epsilon_0 (z^2+a^2)^{3/2}}}$$

{ Note 1- If we take $E = \int dE \cos\theta$ then $\int dl = 2\pi a$ }

* If Total charge in the ring is q then

$$q = \lambda \cdot 2\pi a$$

then $\boxed{E = \frac{qz}{4\pi\epsilon_0 (z^2+a^2)^{3/2}}}$

* Now, find the value of z at which $\vec{E}$ is maximum

$$\frac{dE}{dz} = 0$$

$$\frac{\lambda a}{2\epsilon_0} \frac{d}{dz} \frac{z}{(z^2+a^2)^{3/2}} = 0$$

$$\frac{\lambda a}{2\epsilon_0}\left[\frac{1}{(z^2+a^2)^{3/2}} + z\left(\frac{-3}{\cancel{2}}\right)(z^2+a^2)^{-5/2}(\cancel{2}z)\right] = 0$$

$$\frac{1}{(z^2+a^2)^{3/2}} - \frac{3z^2}{(z^2+a^2)^{5/2}} = 0$$

$$\frac{3z^2}{(z^2+a^2)^{5/2}} = \frac{1}{(z^2+a^2)^{3/2}}$$

$$3z^2 = z^2+a^2$$

$$2z^2 = a^2$$

$$\boxed{z = \pm\frac{a}{\sqrt{2}}}$$

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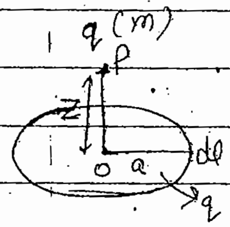
* Now if the distance z is small & put a charge q at the small distance z from the centre of ring.

If total charge = q

then force on it



$$F = \frac{q^2 z}{4\pi\epsilon_0 (z^2+a^2)^{3/2}}$$

$$F \propto z \qquad F = \frac{q^2 z}{4\pi\epsilon_0 a^3} \quad (\text{as } z << a)$$

So its motion will be simple harmonic.

* Find its freq. of oscillation & time period of oscillation

$$F = kz$$

$$\text{So } k = \frac{q^2}{4\pi\epsilon_0 a^3}$$

$$T = 2\pi\sqrt{\frac{m}{k}} \qquad \left\{ \begin{array}{l} \omega = \sqrt{\frac{k}{m}} \\ T = 2\pi/\omega \end{array} \right.$$

$$T = 2\pi\sqrt{\frac{m \cdot 4\pi\epsilon_0 a^3}{q^2}}$$

$$T = \frac{2\pi a}{q}\sqrt{4\pi\epsilon_0 m a}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{q^2}{4\pi\epsilon_0 a^3 m}} = \frac{q}{2a}\sqrt{\frac{1}{\pi\epsilon_0 a m}}$$

* Find $\vec{E}$ at the centre of the ring. (If we put charge at centre)

at centre $z = 0$

So $\vec{E} = 0$

* At far distance from the ring, $z >> a$ then neglect a.

$$\vec{E} = \frac{q}{4\pi\epsilon_0 z^2}$$

At very far distance ring will behave like a point charge.

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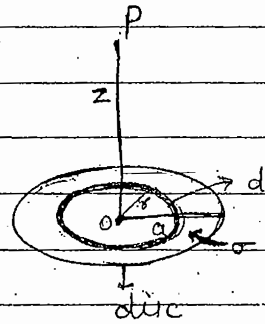
Electric field of a Disc :-

If we have a disc of radius a & surface charge density σ. charge is distributed over surface. We need to find electric field at point P.

If we put cocentric rings (first put ring of radius a & then increasing radius) then it will form a disc.



Suppose a ring of radius r inside the disc,

$$dq = 2\pi r\, dr\, \sigma \qquad (\text{area} = 2\pi r\, dr)$$

Find the field due to this ring,

$$dE = \frac{2\pi r \sigma\, dr \cdot z}{4\pi\epsilon_0 (z^2+r^2)^{3/2}} \qquad \left\{ dE = \frac{qz}{4\pi\epsilon_0 (z^2+a^2)^{3/2}} \right.$$

Total electric field due to a disc

$$E = \frac{z\sigma}{2\epsilon_0}\int_0^a \frac{r\,dr}{(z^2+r^2)^{3/2}}$$

$$= \frac{z\sigma}{2\epsilon_0}\left[-(z^2+r^2)^{-1/2}\right]_0^a$$

$$= \frac{z\sigma}{2\epsilon_0}\left[\left\{\frac{-1}{\sqrt{z^2+a^2}}\right\} + \frac{1}{z}\right]$$

$$\boxed{E = \frac{z\sigma}{2\epsilon_0}\left[\frac{1}{z} - \frac{1}{(z^2+a^2)^{1/2}}\right]}$$

Electric field of a finite disc.

let $z^2+r^2 = t$

$2r\,dr = dt$

$r\,dr = dt/2$

$$\int \frac{r\,dr}{(z^2+r^2)^{3/2}} = \int \frac{dt}{2t^{3/2}}$$

$$= \frac{1}{2}\,\frac{t^{-1/2}}{(-1/2)}$$

$$= -t^{-1/2}$$

(i) Electric field of a infintie disc,

$a \to \infty$

$$\boxed{E = \frac{\sigma}{2\epsilon_0}}$$

This is the same expression obtained by Gauss law.

Note :- If disc is a finite disc then we can not find $\vec{E}$ by Gauss law.

(ii) If z is very small & disc is finite

i.e. $z << a$

$E = ?$

$$(z^2+a^2)^{-1/2} = \frac{1}{a}\left[1+\frac{z^2}{a^2}\right]^{-1/2}$$

$$= \frac{1}{a}\left[1 - \frac{1}{2}\frac{z^2}{a^2}\right]$$

{$(1+x)^n$, cond^n of Binomal expansion is $x<<1$}

$$E = \frac{z\sigma}{2\epsilon_0}\left[\frac{1}{z} - \frac{1}{(z^2+a^2)^{1/2}}\right]$$

Since a is large ↓ neglect

So $$\boxed{E = \frac{\sigma}{2\epsilon_0}}$$

(iii) Disc will behave as a point charge

$z >> a$

$$(z^2+a^2)^{-1/2} = \frac{1}{z}\left[1+\frac{a^2}{z^2}\right]^{-1/2}$$

$$= \frac{1}{z} - \frac{a^2}{2z^3}$$

$$E = \frac{z\sigma}{2\epsilon_0}\left[\frac{1}{z} - \frac{1}{z} + \frac{a^2}{2z^3}\right]$$

$$\boxed{E = \frac{\sigma a^2}{4\epsilon_0 z^2}}$$

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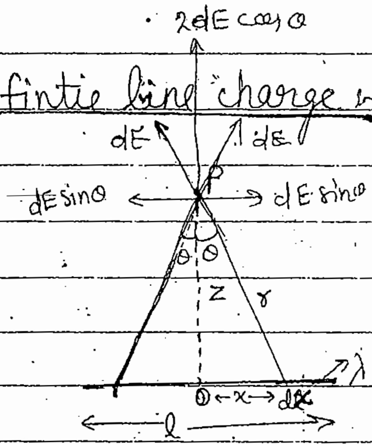
(iv) If Total charge in the disc is q then

$$q = \sigma \cdot \pi a^2$$

so.
$$\boxed{E = \frac{q}{4\pi\epsilon_0 z^2}}$$



Electric field of a finite wire or fintie line charge :-

At a distance z from centre O.

(We can't use Gauss law)

Take a small element dx at a distance x from O.

$$dq = \lambda\, dx$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

$$dE = \frac{\lambda\, dx}{4\pi\epsilon_0 (z^2+x^2)}$$

$$\cos\theta = \frac{z}{r} = \frac{z}{(z^2+x^2)^{1/2}}$$

Total electric field

$$E = \int_0^{l/2} 2\,dE \cos\theta$$

{ If we take $2dE$ then integration will be from 0 to $l/2$.
" " dE " " " " $-l/2$ to $l/2$ }

$$E = \int_0^{l/2} 2 \frac{\lambda\, dx\, z}{4\pi\epsilon_0 (z^2+x^2)^{3/2}}$$

$$= \frac{\lambda z}{2\pi\epsilon_0} \int_0^{l/2} \frac{dx}{(z^2+x^2)^{3/2}}$$

let $x = z\tan\theta$

$dx = z\sec^2\theta\, d\theta$

$$\int \frac{dx}{(z^2+x^2)^{3/2}} = \int \frac{z\sec^2\theta\, d\theta}{z^3\sec^3\theta}$$

$$= \frac{1}{z^2}\int \cos\theta\, d\theta$$

$$= \frac{1}{z^2}\sin\theta$$

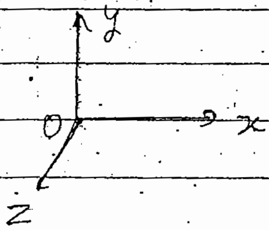
$$\tan\theta = \frac{x}{z}, \quad \sin\theta = \frac{x}{r} = \frac{x}{(z^2+x^2)^{1/2}}$$

$$\int_0^{l/2} \frac{dx}{(z^2+x^2)^{3/2}} = \left[\frac{1}{z^2}\frac{x}{\sqrt{x^2+z^2}}\right]_0^{l/2}$$

$$= \frac{1}{z^2}\left[\frac{l/2}{\sqrt{\frac{l^2}{4}+z^2}} - 0\right]$$

$$\boxed{\vec{E} = \frac{\lambda z}{2\pi\epsilon_0}\frac{1}{z^2}\frac{l/2}{\sqrt{\frac{l^2}{4}+z^2}}\,\hat{y}}$$

dirⁿ of $\vec{E}$ is $+\hat{y}$



Cases :-

(i) If $l \to \infty$

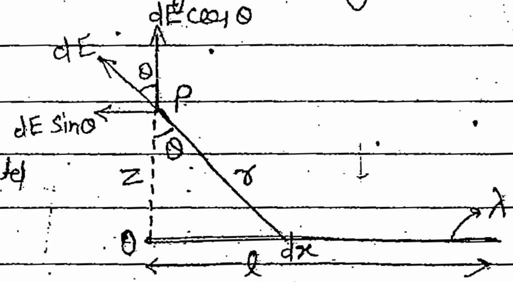
$$E = \frac{\lambda z}{2\pi\epsilon_0}\frac{1}{z^2}\frac{l/2}{l/2\sqrt{1+\frac{4z^2}{l^2}}}$$

$$\boxed{E = \frac{\lambda}{2\pi\epsilon_0 z}}$$

(ii) If $z >> l$

$$E = \frac{\lambda l}{4\pi\epsilon_0 z^2} \Rightarrow \boxed{E = \frac{q}{4\pi\epsilon_0 z^2}}$$

Prob 1 :- Finite wire of length l & charge density λ. Calculate the electric field at P.



$E = \int dE\cos\theta$ i.e.

E along $\hat{y} \Rightarrow E_y$ we have calculated above.

Now,

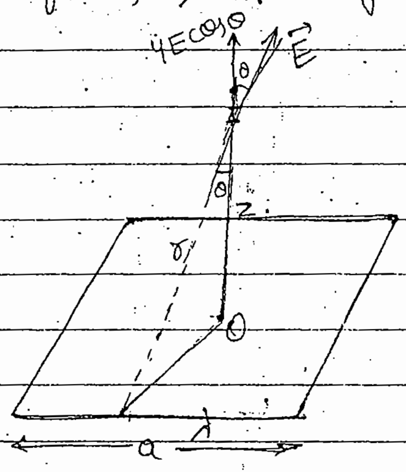
$$E_x = \int dE \sin\theta$$

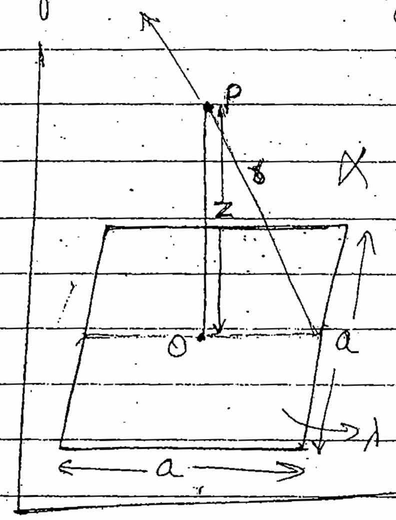
$$= \int_0^l \frac{\lambda\, dx}{4\pi\epsilon_0 (z^2+x^2)^{3/2}}\, x$$

$$E_x = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{-1}{(z^2+a^2)^{1/2}}\right]_0^l$$

$$\boxed{E_x = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{z} - \frac{1}{(z^2+l^2)^{1/2}}\right]}$$

Q4 Find the electric field at a distance z above at the centre of a square loop of side a carring uniform line charge λ.





$\sin\theta$ - Component will cancel out.

Electric field of a finite wire

$$E = \frac{\lambda(l/2)}{2\pi\epsilon_0 z\left(z^2+\frac{l^2}{4}\right)^{1/2}}$$

so for a square loop,

$$E = \frac{\lambda(a/2)}{2\pi\epsilon_0\, r\sqrt{r^2+\frac{a^2}{4}}}$$

$$r = \sqrt{z^2+\frac{a^2}{4}}$$

$$E = \frac{\lambda a}{4\pi\epsilon_0\sqrt{z^2+\frac{a^2}{4}}\sqrt{z^2+\frac{a^2}{2}}}$$

Electric field due to 1 wire along z is

$$E_z = E\cos\theta$$

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$$\cos\theta = \frac{z}{\sqrt{z^2 + a^2/4}}$$

$$E_z = \frac{\lambda a z}{4\pi\epsilon_0 \left(z^2 + \frac{a^2}{4}\right)\sqrt{\left(z^2 + \frac{a^2}{2}\right)}}$$

Total electric field due to 4 wires i.e. for square loop

$$E_z = 4\,\frac{\lambda z a}{4\pi\epsilon_0 \left(z^2 + \frac{a^2}{4}\right)\sqrt{z^2 + \frac{a^2}{2}}}$$

$$\boxed{E_z = \frac{\lambda a z}{\pi\epsilon_0 \left(z^2 + \frac{a^2}{4}\right)\sqrt{z^2 + \frac{a^2}{2}}}}$$ Ans

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electric field = $\frac{}{2\epsilon_0}$

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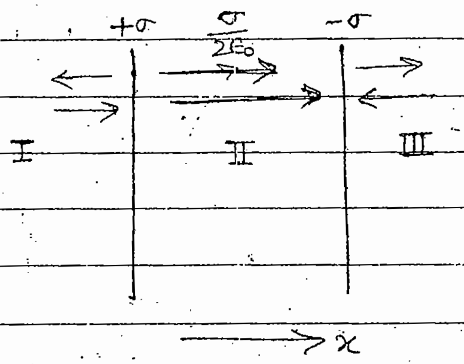
Que 1 ① If there are two plane sheets having charge density σ & $-\sigma$ respectively then find out $\vec{E}$ in I, II & III region.



Arrow shows the dir^n of Electric field
I. Draw the arrows for $+\sigma$ sheet
II. for $-\sigma$ sheet & so on

Magnitude of electric field $= \frac{\sigma}{2\epsilon_0}$

$\vec{E}$ in region I $\rightarrow$ 0

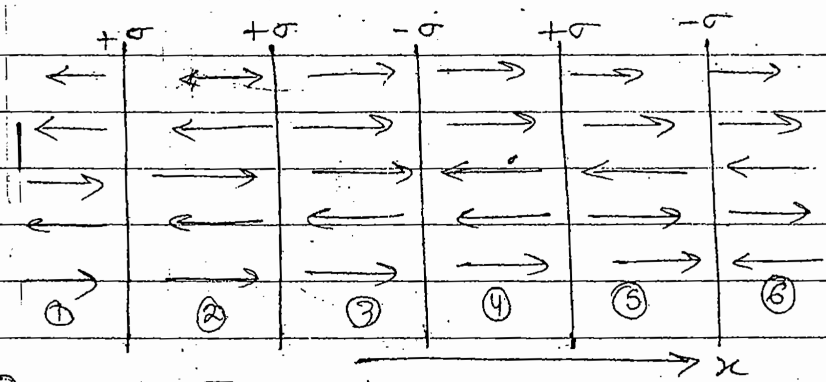
II $\rightarrow$ $\frac{\sigma}{\epsilon_0}$ $\left\{ \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0} \right.$

III $\rightarrow$ 0

Dir^n of $\vec{E}$ is $+\hat{x}$

* Dir^n of $\vec{E}$ is always from + to −.

Que 2



Region ① $\rightarrow E = \frac{\sigma}{2\epsilon_0}$ $(-\hat{x})$

(2) $\rightarrow E = \frac{\sigma}{2\epsilon_0}$ $(+\hat{x})$

(3) $\rightarrow E = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{3\sigma}{2\epsilon_0}$ $(+\hat{x})$

(4) $\rightarrow E = \frac{\sigma}{2\epsilon_0}$ $(+\hat{x})$

(5) $\rightarrow E = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{3\sigma}{2\epsilon_0}$ $(+\hat{x})$

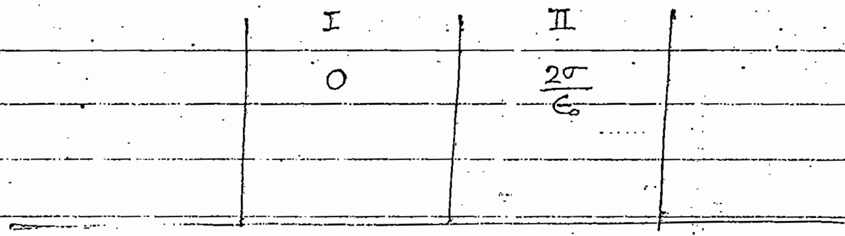
(6) $\rightarrow E = \frac{\sigma}{2\epsilon_0}$ $(+\hat{x})$

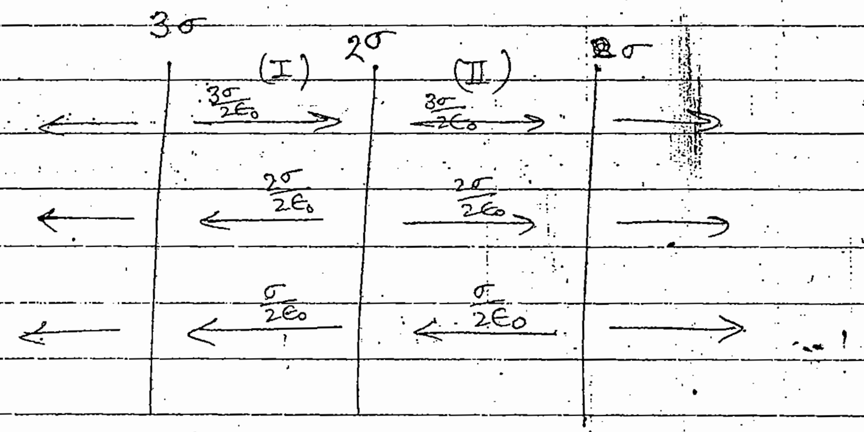
Note :- Dir^n of Electric field

for $+q \rightarrow$ away

$-q \rightarrow$ towards

Q.2 – Three infinite non-conducting sheets with uniform surface charge densities, σ, 2σ & 3σ are arrange to be parallel. What is there order from left to right if electric field $\vec{E}$ produce by the arrangement has the magnitude, $E = 0$ in one region & $E = \frac{2\sigma}{\epsilon_0}$ in the another region.





In I region, $\vec{E} = \frac{3\sigma}{2\epsilon_0}\hat{x} - \frac{2\sigma}{2\epsilon_0}\hat{x} - \frac{\sigma}{2\epsilon_0}\hat{x}$

$$\vec{E} = \frac{3\sigma}{2\epsilon_0}\hat{x} - \frac{3\sigma}{2\epsilon_0}\hat{x} \Rightarrow \boxed{\vec{E} = 0}$$

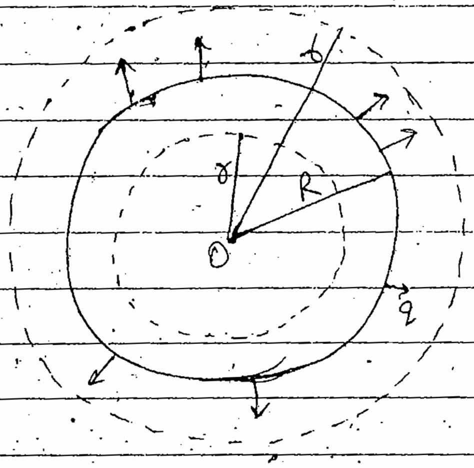
In II Region, $\vec{E} = \frac{3\sigma}{2\epsilon_0}\hat{x} + \frac{2\sigma}{2\epsilon_0}\hat{x} - \frac{\sigma}{2\epsilon_0}\hat{x}$

$$= \frac{5\sigma}{2\epsilon_0}\hat{x} - \frac{\sigma}{2\epsilon_0}\hat{x} = \frac{4\sigma}{2\epsilon_0}\hat{x}$$

$$\boxed{\vec{E} = \frac{2\sigma}{\epsilon_0}\hat{x}}$$

Conducting Surfaces :-

Q.1 :- Find the electric field inside & outside a conducting sphere of radius R and carrying total charge q.



Inside :-

charge enclosed

$$q_{enc} = 0$$

$$\vec{E} = 0$$

Electric field inside the conducting material is zero. No static charge can remain inside the conducting material.

Volume charge density for conducting material is zero.

Properties of conducting sphere,

(i) $\vec{E} = 0$

(ii) $\rho = 0$

Outside :- charge enclosed = q

$$E \cdot 4\pi r^2 = \frac{q}{\epsilon_0}$$

$$\boxed{\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}}$$

(iii) Electric field $\vec{E}$ is normal to the conducting surface.

✓(iv) Conductors are equipotential surface i.e. at every point of the conductor, potential is same.

for reference, $\boxed{\text{Pot}^n \text{ of earth} = 0}$ bcoz if we apply some charge, pot^n of earth does not change. It remain constant. ~~Pot~~ That's why for reference, we consider pot^n of earth zero. Since capacitance of earth is very larger so its pot^n does not change. If we connect sphere to earth then it will take the pot^n of earth.

* Only free charge goes into the ground. Induced charge never goes in the ground.

→ Similarly, ~~for~~
If we have a conducting cylender then

Electric field inside the cylender $\vec{E} = 0$
" " outside " " $\vec{E}$ normal to the surface.

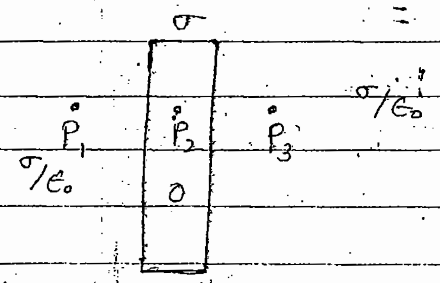
$$E_{out} = \frac{KR^3}{3\epsilon_0 s}\hat{s}$$

Conducting Plates :-

We have a plate made up of conducting ~~sphere~~ surface, having charge denside σ.

(i) at P_2, $\vec{E}_{in} = 0$

(ii) at P_1 & P_2, $\vec{E} = \frac{\sigma}{\epsilon_0}$



Conducting plate have two surface ~~wh~~ even if it is very thin.

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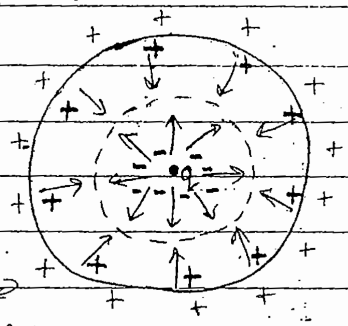
→ Why $\vec{E}$ is zero inside the conducting material.

* In conductors, there are free e^-.
In Insulator, there are not present free e^-.
This is the difference b/w conductor & insulator.

\# If we take a neutral conducting sphere then total $q_{enc} = 0$ so $E_{in} = 0$ bcoz no. of free e^- = +ve charges created when e^- become free.

\# If we take a conducting sphere and charge q is placed at centre.

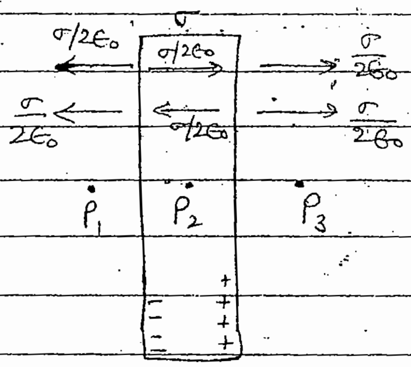


then still $q_{enc} = 0$
bcoz dir^n of $\vec{E}$ created by q at centre is away from charge. Due to +q, inward force, $F = -eE$. Due to free force apply on free e^-, they become free & +ve charge create over the surface. Dir^n of $\vec{E}$ by +ve charge is ~~towards q (inside)~~ (from + to −)

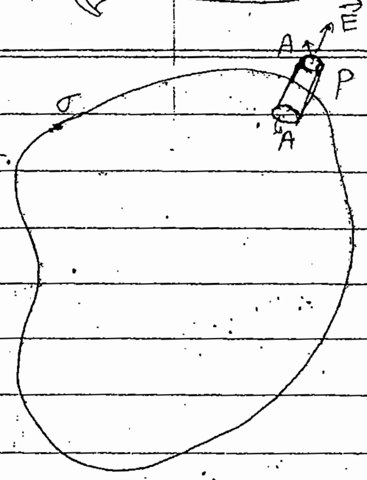
→ charge outward force. $F = qE$

So Total $q_{enc} = 0$
So $E_{in} = 0$ (due to charge seperation)
but outside E_{out} = Non zero.

⇒ field due to I surface push ~~create~~ −ve charge on I & +ve charge on III surface.
Again field due to II surface push −ve charges on II surface. So Net $\vec{E}$ at P_2 is 0. $\boxed{\vec{E} = 0}$



Arbitrary Surfaces:- If we take a arbitrary surface of charge density σ.



- Find out $\vec{E}$ at point P which is just outside the conductor. What is the total flux inside the conductor.

If area is A. $\boxed{charge = \sigma A}$.

The flux passing through the curved part is zero ($\vec{E}$ is $\perp$ to the surface) The flux passing through the lower cross section which is inside the surface is also zero & flux passing through upper cross section is Non zero.

Electric field inside the conductor $E = 0$

" " " outside " " is

$$\oint E \cdot dS = \frac{q}{\epsilon_0} \Rightarrow EA = \frac{\sigma A}{\epsilon_0}$$

$$\boxed{E = \frac{\sigma}{\epsilon_0}}$$

$$\boxed{\vec{E} = \frac{\sigma}{\epsilon_0} \hat{n}}$$

- Electrostatic field is conservative field i.e. work done by the field is path independent

Electrostatic field is

$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^2} \hat{r}$$

$$\boxed{\vec{\nabla} \times \vec{E} = 0} \rightarrow \text{differential form}$$

$$\boxed{\oint_c E \cdot dl = 0} \rightarrow \text{Integral form}$$

$$\left\{ \nabla \times E = 0 \Rightarrow \int (\nabla \times \vec{E}) \cdot d\vec{S} = 0 \Rightarrow \oint_c \vec{E} \cdot d\vec{l} = 0 \right\}$$

{for conservative field, curl of field should be ... }

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$E \cdot dl \rightarrow$ work done for a unit test charge.

$\left\{ E = \frac{F}{q} \right\}$ & $\{ W = F \cdot dl \}$

$\left\{ \text{for } q = 1, \quad E = F, \quad E = \frac{W}{dl}, \quad \cancel{E \cdot dl = W} \right\}$

Que :- $\vec{E} = k(xy\,\hat{x} + 2yz\,\hat{y} + 3xz\,\hat{z})$

$\vec{E} = k(y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\,\hat{z})$

Find wheather these two electric fields are possible electrostatic fields.

OR

Which one of the following is impossible electrostatic field.

(i) $\vec{E} = k(xy\,\hat{x} + 2yz\,\hat{y} + 3xz\,\hat{z})$

$$\nabla \times \vec{E} = k \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix}$$

$$= k\left[\hat{x}(0 - 2y) - \hat{y}(3z - 0) + \hat{z}(0 - x)\right]$$

$$= k\left[-2y\,\hat{x} - 3z\,\hat{y} - x\,\hat{z}\right]$$

$\nabla \times E \neq 0$

(ii) $\vec{E} = k[y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\,\hat{z}]$

$$\nabla \times \vec{E} = k \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2yz \end{vmatrix}$$

$$= k\left[\hat{x}(2z - 2z) - \hat{y}(0 - 0) + \hat{z}(2y - 2y)\right]$$

$$= k\left[0\hat{x} + 0\hat{y} + 0\hat{z}\right] = 0$$

(i) Impossible Electric field.

(ii) Possible Electric field

$\nabla \times E = 0$ (only in Electrostatic)
$\nabla \times E \neq 0$ (Non electrostatic)

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Que :- If $\vec{\nabla} \times \vec{E} = 0$

$\bar{E} = -\bar{\nabla} V$

where

$V \rightarrow$ Scalar Pot^n

It is electrostatic when it does not depand on t.
If V depands on t then $\rightarrow$ Non electrostatic.

We can find V by E $\Rightarrow$ $\boxed{V = -\int_0^r E \cdot dr}$

O $\rightarrow$ reference point

We know that

$$\text{Pot}^n \; V = \frac{W}{q}$$

Pot^n is work done per unit charge.

Pot^n at ∞ (ref. point) is zero. Pot.^{n} of any finite point V w.r. to ∞ is pot^n. At ∞, Pot^n is 0.
Pot^n of any finite point w.r. to a finite point is pot^n difference.

V_a V_b

$V = 0 \rightarrow$ Pot^n
$V_a - V_b \neq 0 \rightarrow$ Pot^n diff.

In Pot^n Energy we take a ~~finite~~ unity test charge.
for unity test charge, pot^n is the pot^n energy itself.

$$U = qV$$

Work done on a system is stored in terms of energy.
If work is done on unity test charge then pot^n engy is = pot^n.
& if work is done on finite test charge then (i.e. work done to move a finite charge from ∞ to a point) is stored as pot^n energy.

We generally take reference point from $0 \rightarrow \infty$.

In all cases we take ∞ as a reference point except where charge itself extended to ∞, In this case we take a finite ref. point.

e.g. line charge

| it extended to itself to ∞. So we can't take ∞ as ref. point.

For point charge, $V = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r}$

for line charge $V = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda dl}{r}$

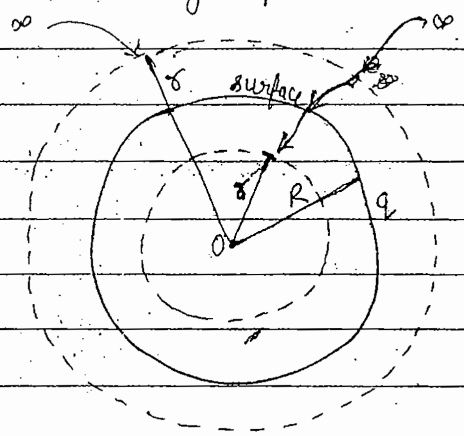
Surface charge $V = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma da}{r}$

Volume charge $V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho d\tau}{r}$

→ Where symmetry is present, I find electric field & then potn.

→ In Non symmetry cases → I find potn & then electric field.

Que - Find the potn inside & outside a conducting sphere of radius R & charge q.



Potn $V_{out} = -\int \vec{E} \cdot d\vec{r}$

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Outside electric field $\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$

charge is not extended to ∞ itself so we'll tak ∞ as reference point.

Work done required to take a unity test charge from ∞ to r is pot^n.

$$V_{out} = -\int_{\infty}^{r} E \cdot dr$$

$$= -\int_{\infty}^{r} \frac{q}{4\pi\epsilon_0 r^2} \cdot dr$$

$$= -\frac{q}{4\pi\epsilon_0}\left[-\frac{1}{r}\right]_{\infty}^{r}$$

$$\boxed{V_{out} = \frac{q}{4\pi\epsilon_0 r}}$$

To find Vin → Work done to take a unity test charge from ∞ to surface & surface to r.

$$V_{in} = -\int_{\infty}^{r} E \cdot dr$$

$$= -\int_{\infty}^{R} \frac{q}{4\pi\epsilon_0 r^2} dr - \int_{R}^{r} 0 \; dr$$

$$\boxed{V_{in} = \frac{q}{4\pi\epsilon_0 R}}$$

$W = F \cdot dr$

for unity test charge $E = F$ bcoz $E = \frac{F}{q}$

then work done to take the unity test charge from surface to r is 0.

$$\boxed{V_{centre} = \frac{q}{4\pi\epsilon_0 R}}$$

→ Pot^n inside the conductor is equal to the pot^n at centre.

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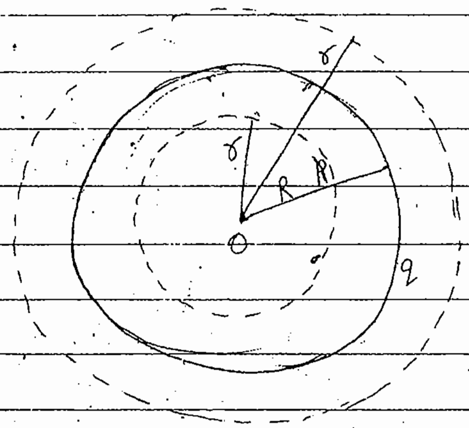
Qn :- Find the potⁿ inside & outside of a uniformly charged sphere having radius R & total charge q.



Uniformly charged means ρ is constant throughout.

$$\vec{E}_{out} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$$

$$\vec{E}_{in} = \frac{qr}{4\pi\epsilon_0 R^3}\hat{r}$$

$$V_{out} = -\int_{\infty}^{r} \vec{E}\cdot d\vec{r} = -\int_{\infty}^{r} \frac{q}{4\pi\epsilon_0 r^2}dr = \frac{-q}{4\pi\epsilon_0}\left[\frac{-1}{r}\right]_{\infty}^{r}$$

$$\boxed{V_{out} = \frac{q}{4\pi\epsilon_0 r}}$$

$$V_{in} = -\int_{\infty}^{R} \frac{q}{4\pi\epsilon_0 r^2}dr - \int_{R}^{r} \frac{qr}{4\pi\epsilon_0 R^3}dr$$

$$= -\frac{q}{4\pi\epsilon_0}\left[\frac{-1}{r}\right]_{\infty}^{R} - \frac{q}{4\pi\epsilon_0 R^3}\left[\frac{r^2}{2}\right]_{R}^{r}$$

$$= \frac{q}{4\pi\epsilon_0 R} - \frac{q}{4\pi\epsilon_0 R^3}\left[\frac{r^2}{2} - \frac{R^2}{2}\right]$$

$$= \frac{q}{4\pi\epsilon_0 R}\left[1 - \frac{r^2}{2R^2} + \frac{1}{2}\right]$$

$$= \frac{q}{4\pi\epsilon_0 R}\left[\frac{2R^2 - r^2 + R^2}{2R^2}\right]$$

$$\boxed{V_{in} = \frac{q}{8\pi\epsilon_0 R^3}(3R^2 - r^2)}$$

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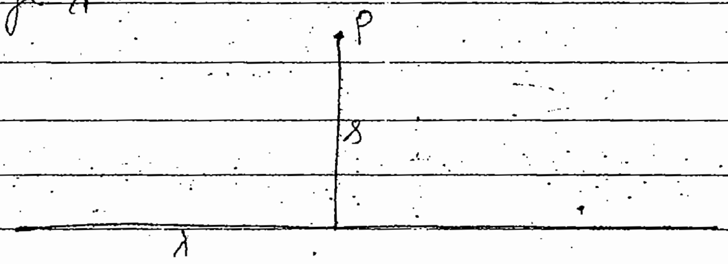
$$\boxed{V_{centre} = \frac{3q}{8\pi\epsilon_0 R}}$$ (at centre $r = 0$)

$$\boxed{V_{centre}\left(\begin{matrix}\text{uniformly}\\ \text{charged}\end{matrix}\right) = \frac{3}{2}\, V_{centre}\,(\text{conducting})}$$

Que :- Find the potⁿ at a distance s from an infinitely long straight wire that carries uniform line charge λ.



$\{\cancel{q = \lambda s}\}$

at P, $\vec{E} = \dfrac{\lambda}{2\pi\epsilon_0 s}\,\hat{s}$

$$V = -\int_a^s E \cdot ds$$

{a → finite point at which we know the potⁿ}

$$= -\int_a^s \frac{\lambda}{2\pi\epsilon_0 s}\, ds$$

$$= \frac{-\lambda}{2\pi\epsilon_0}\left(\ln s\right)_a^s = \frac{-\lambda}{2\pi\epsilon_0}\left(\ln\frac{s}{a}\right)$$

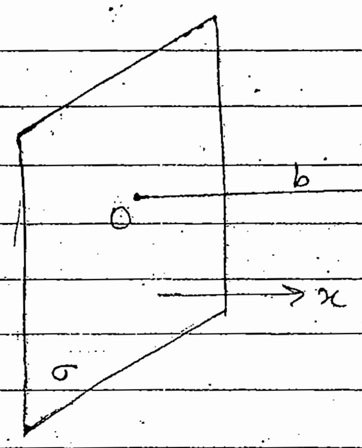
$$\boxed{V = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{a}{s}\right)}$$

a → reference point

s → at which we want to find potⁿ.

Que :- find the potⁿ at a distance b from an infinite sheet of charge σ

→ Sheet is extended to itself ∞ so ref. point can not be ∞.



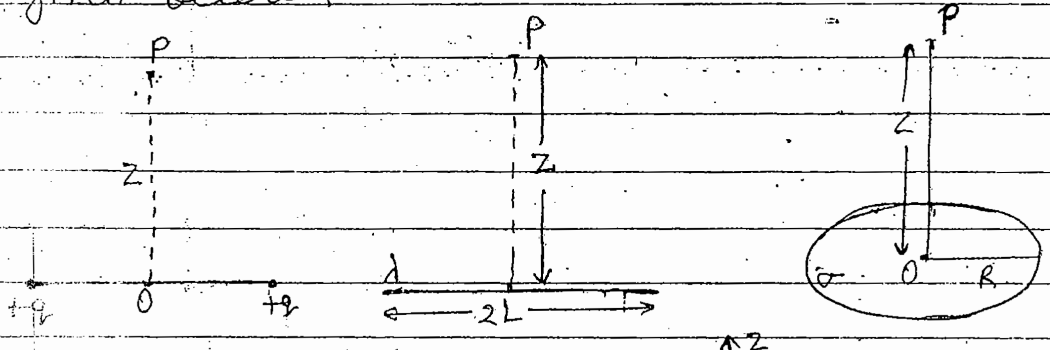
at P, $\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{n}$

$$V = -\int_a^b \frac{\sigma}{2\epsilon_0}\,dx$$ a → reference point

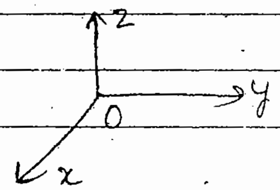
$$= -\frac{\sigma}{2\epsilon_0}[x]_a^b = -\frac{\sigma}{2\epsilon_0}[b-a]$$

$$\boxed{V = \frac{\sigma}{2\epsilon_0}(a-b)}$$

Q.1:- Calculate the potⁿ of 3 charge configurations given below:



Co-ordinate system:-



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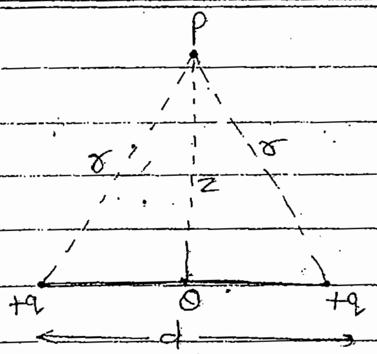
(i)



$$V_2 = V_1 = \frac{q}{4\pi\epsilon_0 r}$$

$$= \frac{q}{4\pi\epsilon_0 \sqrt{z^2 + \frac{d^2}{4}}}$$

Pot^n is a scalar quantity. Total pot^n at P will be

$$V = V_1 + V_2$$

$$V = \frac{2q}{4\pi\epsilon_0\sqrt{z^2+\frac{d^2}{4}}} \Rightarrow \boxed{V = \frac{q}{2\pi\epsilon_0\sqrt{z^2+\frac{d^2}{4}}}}$$

Electric field at point P will be

$$\vec{E} = -\vec{\nabla} V$$

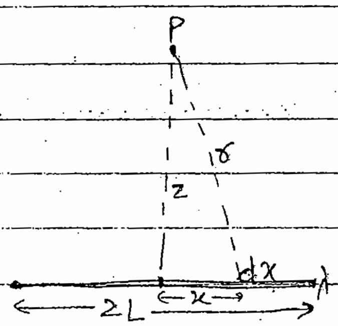
$$= -\frac{\partial V}{\partial z}\hat{z}$$

$$= -\frac{q}{2\pi\epsilon_0} \cdot -\frac{1}{2}\left(z^2+\frac{d^2}{4}\right)^{-3/2}(2z)$$

$$\boxed{\vec{E} = \frac{qz}{2\pi\epsilon_0\left(z^2+\frac{d^2}{4}\right)^{3/2}}\hat{z}}$$

(ii)

$$r = \sqrt{z^2+x^2}, \quad V = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda dl}{r}$$



$$V = \frac{\lambda}{4\pi\epsilon_0}\int_0^L \frac{\lambda dx}{\sqrt{z^2+x^2}}$$

$$V = \frac{\lambda}{2\pi\epsilon_0}\ln\left[x+\sqrt{z^2+x^2}\right]_0^L$$

$$\left\{\text{Using } \int\frac{dx}{(a^2+x^2)^{1/2}} = \ln\left[x+\sqrt{a^2+x^2}\right]\right\}$$

$$V = \frac{\lambda}{2\pi\epsilon_0}\ln\left[\frac{L+\sqrt{z^2+L^2}}{z}\right]$$

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electric field $\vec{E} = -\frac{\partial V}{\partial z}\hat{z}$

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{\frac{L+\sqrt{L^2+z^2}}{z}}\left\{\frac{z\frac{1}{2}(z^2+L^2)^{-1/2} - (\sqrt{z^2+L^2}+L)}{z^2}\right\}\hat{z}$$

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{L+\sqrt{L^2+z^2}}\left[\frac{1}{2\sqrt{z^2+L^2}} - \frac{(\sqrt{z^2+L^2}+L)}{z^2}\right]\hat{z}$$

$$= \frac{\lambda}{2\pi\epsilon_0}\frac{1}{L+\sqrt{L^2+z^2}}\left[\frac{z^2-2z^2-2L^2}{2\sqrt{z^2+L^2}\,z^2}\right]\hat{z}$$

$$= \frac{\lambda}{2\pi\epsilon_0(L+\sqrt{L^2+z^2})}\frac{z^2-2L^2}{2\sqrt{z^2+L^2}\,z^2}\hat{z}$$

$$= \frac{\lambda}{4\pi\epsilon_0}$$

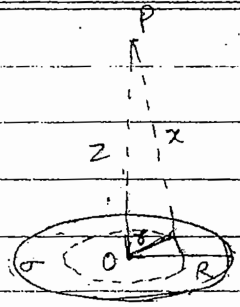
$$\vec{E} = -\frac{\partial}{\partial z}\left[\frac{\lambda}{2\pi\epsilon_0}\left(\ln\{L+\sqrt{z^2+L^2}\} - \ln z\right)\right]$$

$$= -\frac{\lambda}{2\pi\epsilon_0}\left[\frac{2z}{(L+\sqrt{L^2+z^2})\,2\sqrt{z^2+L^2}} - \frac{1}{z}\right]\hat{z}$$

$$= \frac{-\lambda}{2\pi\epsilon_0}\left[\frac{z}{(L+\sqrt{z^2+L^2})\sqrt{z^2+L^2}} - \frac{1}{z}\right]\hat{z}$$

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0}\left[\frac{1}{z} - \frac{z}{(L\sqrt{z^2+L^2}+z^2+L^2)}\right]\hat{z}$$

(iii)



$$dq = 2\pi r\, dr\, \sigma$$

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{\sqrt{z^2+r^2}}$$

$$x = \sqrt{z^2+r^2}$$

$$V = \frac{\sigma}{4\pi\epsilon_0}\int_0^R \frac{2\pi r\, dr}{(z^2+r^2)^{1/2}}$$

$$z^2+r^2 = t$$

$$2r\,dr = dt$$

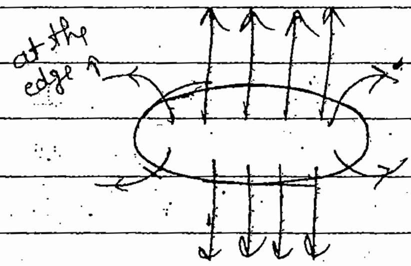
$$V = \frac{\sigma \cdot 2\pi}{4\pi\epsilon_0}\int \frac{dt}{2(t)^{1/2}}$$

$$V = \frac{\sigma}{4\epsilon_0}\left[\frac{t^{1/2}}{1/2}\right]$$

$$= \frac{\sigma}{2\epsilon_0}\left[(z^2+r^2)^{1/2}\right]_0^R \Rightarrow \boxed{V = \frac{\sigma}{2\epsilon_0}\left[(z^2+R^2)^{1/2} - z\right]}$$

$$\boxed{V_{centre} = \frac{\sigma R}{2\epsilon_0}} \quad \{\text{at Centre} \rightarrow z = 0\}$$

<u>Potential at the edge</u>,



$$\boxed{V_{edge} = \frac{\sigma R}{\pi\epsilon_0}}$$

$\therefore$ $\boxed{V_{centre} > V_{edge}}$ Due to <u>Edge Effect</u>.

Electric field lines at the edges are not entirely normal to the surface.

Hence V_{edge} is different from the V_{centre}.

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Qu 1- Find whether the Electric field $\vec{E} = (12x^2 - y^2)\hat{x} - 2xy\hat{y}$ is conservative or not. If conservative find the potential as a func^n of position.

$$\vec{E} = (12x^2 - y^2)\hat{x} - 2xy\,\hat{y}$$

$\vec{E}$ will be conservative if $\vec{\nabla} \times \vec{E} = 0$

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (12x^2 - y^2) & -2xy & 0 \end{vmatrix}$$

$$= \hat{x}[0] - \hat{y}[0] + \hat{z}[-2y + 2y]$$

$$\vec{\nabla} \times \vec{E} = 0$$

so $\vec{E}$ is conservative

Now find $V(r) = ?$

$$\vec{E} = -\vec{\nabla} V \Rightarrow V = -\int E \cdot dr$$

$$= -\left[\int (12x^2 - y^2)\,dx - \int 2xy\,dy\right]$$

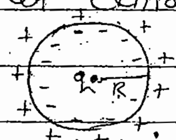
$$\left\{ \cancel{= -\left[12\frac{x^3}{3} + xy^2 - 2x\frac{y^2}{2}\right]} \right.$$
$$\left. \cancel{V = -4x^3 + 2xy^2} \right\}$$

$$V = -\int d(4x^3 - xy^2)$$

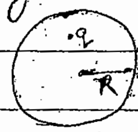
$$\boxed{V = xy^2 - 4x^3 + C} \text{ Ans}$$

If we have a conductor & $+q$ is place at centre of the conductor. So at centre

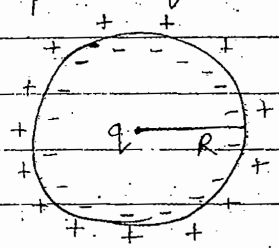
$\boxed{q_{inside} = 0}$ & $\rho = 0$



If instead of centre, we place charge q anywhere else then still, $\rho_{inside} = 0$

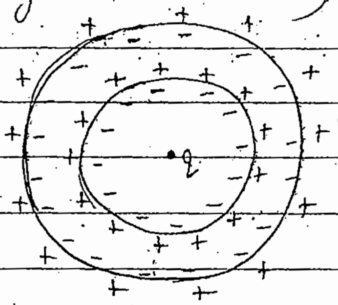


If we have a spherical shell or hollow conducting sphere of radius R then



On Inner surface charge will be $-q$ & on outer surface $+q$. (+ve charge is uniformly distributed)

If we have further place the conducting shell.



Que - A spherical conductor of radius 'a' has a charge $+q$ placed on it. This is surrounded by thin spherical conducting shell of radius b. It is connected to the earth. Find

(a) charges on outer & inner surface of the shell.

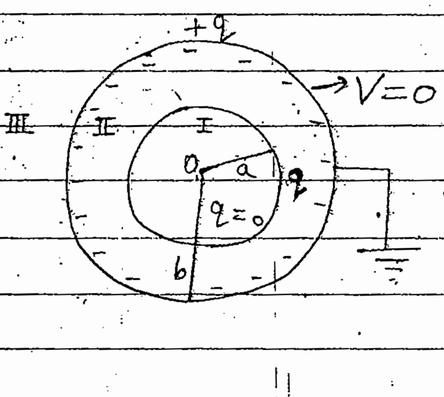
(b) pot^n in region I, II & III.

(c) The electric field in region I, II & III.

Conductors are equipot^n surface.

It is not necessary that - if $pot^n = 0$ then $q = 0$

So only satisfy the pot^n condition.



for conducting sphere of radius a, inner charge

$$\boxed{q = 0}$$

On Inner surface of conducting sphere of radius b the -ve charge is uniformly distributed & on outer surface $q = ?$

It is grounded so $V = 0$

At Outer surface

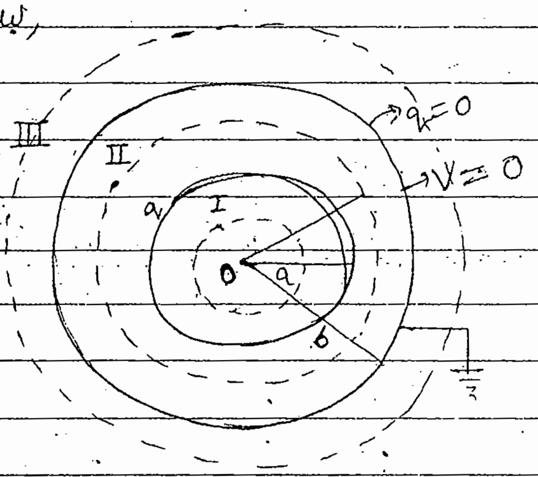
$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{b} - \frac{1}{4\pi\epsilon_0}\frac{q}{b} + \frac{1}{4\pi\epsilon_0}\frac{q}{b} = 0$$

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{b} = 0$$

$$\Rightarrow \boxed{q = 0}$$

So charge is zero at outer surface.

(b) Now find $\vec{E}$ using Gauss law,



Region I,

$$\boxed{E_I = 0}$$

Region II,

$$\boxed{E_{II} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}}$$

Region III,

$$\boxed{E_{III} = 0}$$

(c) Potential :- $V_{III} = -\int_\infty^r E_{III}\cdot dr = 0$

$$\boxed{V_{III} = 0}$$

$$V_{II} = -\int_\infty^b E_{III}\, dr - \int_b^r \frac{q}{4\pi\epsilon_0 r^2}\, dr$$

$$= 0 + \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r}\right]_b^r$$

$$\boxed{V_{II} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{1}{b}\right]}$$

Conducting sphere shield the inner charge from outer charge but
Not " " outer " " inner "

$$V_I = -\int_{\infty}^{r} \vec{E}\cdot d\vec{r}$$

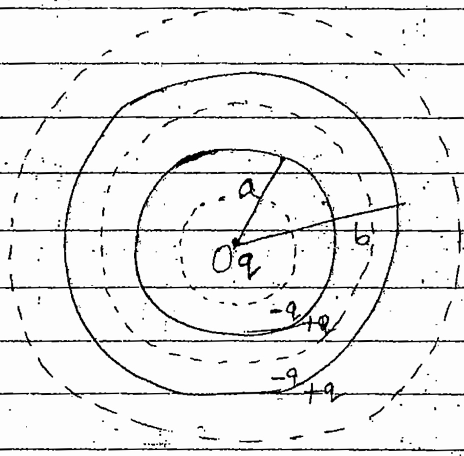
$$= -\int_{\infty}^{b} E_{III}\cdot dr - \int_{b}^{a} \frac{q}{4\pi\epsilon_0 r^2}dr - \int_{a}^{r} E_I \cdot dr$$

$$= 0 + \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r}\right]_b^a - 0$$

$$\boxed{V_I = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{a}-\frac{1}{b}\right]}$$

Que: If we have 2 conducting spherical shells of radius a & b. find
(i) distribution of charge for both surface
(ii) potⁿ V for $r<a$, $a<r<b$, $r>b$
(iii) $\vec{E}$ " " "

(i) inner $\rightarrow -q$
outer $\rightarrow +q$



(iii) Net charge $\rightarrow +q$
In all III regions,
behaviour of $\vec{E}$ is same
for $r<a$, $a<r<b$, $r>b$

$$\boxed{E = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}}$$

(ii) for $r>b$

$$V = -\int_{\infty}^{r} E\cdot dr \Rightarrow V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

for $a<r<b$,

$$V = -\int_{\infty}^{b} E\cdot dr - \int_{b}^{r} E\cdot dr = +\frac{1}{4\pi\epsilon_0}\frac{q}{b} + \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r}-\frac{1}{b}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

$r<a$ = $V = -\int_{\infty}^{r} E\cdot dr = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$

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Note 1 – If all the charges are inside then ~~cal~~ take sum of all the charges & distance where we calculate the potⁿ i.e.

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

[for precious figure]

If some charge are inside & some outside then
for inside charges → takes distance where we calculate potⁿ
for outside " → " " where these charges are placed.

If outer charge is +q & −q then

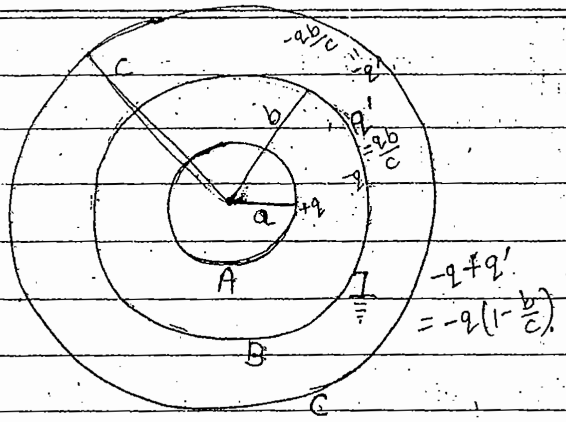
$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{b} - \frac{1}{4\pi\epsilon_0}\frac{q}{b} = 0$$

Note 1 – "To find the potⁿ in spherical symmetric cases place all the inside charges at the distance where we are calculating the potⁿ & all outside charges must be placed at the distance where they are."

This is the direct method.

Ques: – 3 cocentric spherical shells A, B & C of radii a, b & c respectively. The shell A is given a charge +q & shell C, −q. Shell B is grounded. Find the charges & potentials on the surfaces of A, B & C.

charge on B can not be exactely grounded due to shell C. & Also can not be exactely equal to q. Let ~~ch~~ induced charge on shell B is q′.
i.e. At B shell, potⁿ = 0
but charge may or may not be zero.



Pot^n At B,

$$V_B = \frac{q'}{4\pi\epsilon_0 b} + \frac{q}{4\pi\epsilon_0 b} - \frac{q}{4\pi\epsilon_0 b} - \frac{q}{4\pi\epsilon_0 c} - \frac{q'}{4\pi\epsilon_0 c} + \frac{q'}{4\pi\epsilon_0 c}$$

$$V_B = \frac{q'}{4\pi\epsilon_0 b} - \frac{q}{4\pi\epsilon_0 c} = 0 \quad \Big\} \text{At B, } V=0$$

$$\frac{q'}{b} - \frac{q}{c} = 0$$

$$\Rightarrow \boxed{q' = \frac{qb}{c}}$$

We know that $\boxed{V_B = 0}$

$$V_A = \frac{q}{4\pi\epsilon_0 a} - \frac{q}{4\pi\epsilon_0 b} + \frac{q\cancel{b}}{4\pi\cancel{\epsilon_0}\cancel{b}c} - \frac{qb}{4\pi\cancel{\epsilon_0} c\cdot c} - \frac{q}{\cancel{4\pi\epsilon_0} c} + \frac{q\cancel{b}}{c\,\cancel{4\pi}\epsilon_0 c}$$

$$\boxed{V_A = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]}$$

$$V_C = \frac{q}{\cancel{4\pi\epsilon_0} c} - \frac{q}{\cancel{4\pi}\epsilon_0 c} + \frac{q\cancel{b}}{c\,\cancel{4\pi}\epsilon_0 c} - \frac{q\cancel{b}}{c\,4\pi\epsilon_0 c} - \frac{q}{4\pi\epsilon_0 c} + \frac{qb}{c\,4\pi\epsilon_0 c}$$

$$\boxed{V_c = \frac{-q}{4\pi\epsilon_0 c} + \frac{qb}{4\pi\epsilon_0 c^2}}$$

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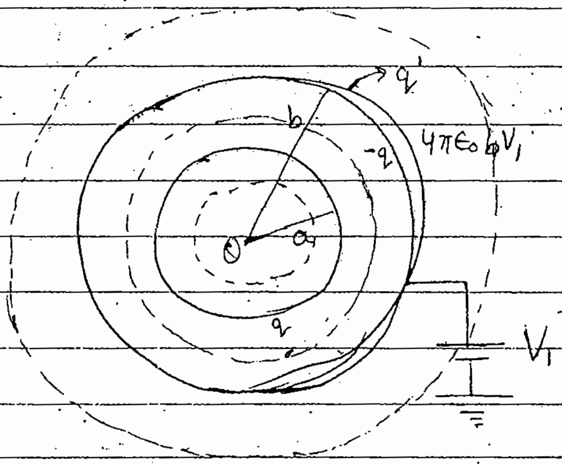
Ques :- A spherical conductor of radius 'a' has a charge q placed on it. This is surrounded by a thin spherical shell of radius b & connected to the earth through a battery of pot^n V_1.

(a) Find the charges on the outer & inner surface of shell.

(b) Find the pot^{n}s & Electric fields at a distance r from the centre of the sphere where
(i) $r < a$
(ii) $a < r < b$
(iii) $r > b$



Let charge on outer surface is q'.
On outer surface pot^n is V_1

$$V_1 = \cancel{\frac{q}{4\pi\epsilon_0 b}} - \cancel{\frac{q}{4\pi\epsilon_0 b}} + \frac{q'}{4\pi\epsilon_0 b}$$

$$\Rightarrow \boxed{q' = 4\pi\epsilon_0 bV_1}$$

On Inner surface → $-q$

(b) Pot^n :-

(i) ~~...~~ $r > b$

$$V = \frac{q}{4\pi\epsilon_0 r} - \frac{q}{4\pi\epsilon_0 r} + \frac{4\pi\epsilon_0 bV_1}{4\pi\epsilon_0 r}$$

$$\boxed{V = \frac{bV_1}{r}}$$

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(ii) $a < r < b$

$$V = \frac{q}{4\pi\epsilon_0 r} - \frac{q}{4\pi\epsilon_0 b} + \frac{4\pi\epsilon_0 b V_1}{4\pi\epsilon_0 b}$$

$$\boxed{V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{1}{b}\right] + V_1}$$

(iii) $r < a$

$$V = \frac{q}{4\pi\epsilon_0 a} - \frac{q}{4\pi\epsilon_0 b} + \frac{4\pi\epsilon_0 b V_1}{4\pi\epsilon_0 b}$$

$$\boxed{V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right] + V_1}$$

Electric field

i) $r > b$,

either $\vec{E} = -\vec{\nabla} V$ or by Gauss law.

$$\cancel{V = -\int_\infty^r \vec{E}\cdot d\vec{r}} = \cancel{-\int_\infty^r \frac{bV_1}{r}}\, dr,$$

$$\cancel{V = -bV_1[\log r]_\infty^r}$$

$$\oint E \cdot ds = \frac{q}{\epsilon_0}$$

$$E \cdot 4\pi r^2 = \frac{4\pi\epsilon_0 b V_1}{\epsilon_0}$$

$$\boxed{\vec{E} = \frac{bV_1}{r^2}\hat{r}}$$

{ by gradient $E = -\vec{\nabla} V = -\vec{\nabla}\left(\frac{bV_1}{r}\right) = \frac{bV_1 \vec{r}}{r^3} = \frac{bV_1}{r^2}\hat{r}$

$\because r = \sqrt{x^2+y^2+z^2}$

(ii) $a < r < b$,

$$\oint E \cdot ds = \frac{q}{\epsilon_0}$$

$$E \cdot 4\pi r^2 = \frac{q}{\epsilon_0}$$

$$\boxed{\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}}$$

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$r < a$,

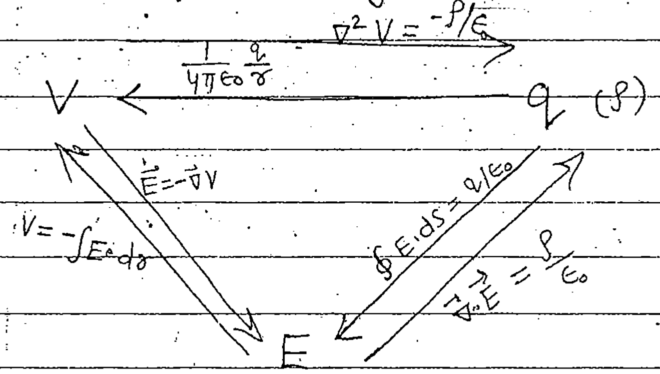
(iii) $q_{enc} = 0$

$$\oint E \cdot ds = \frac{q}{\epsilon_0}$$

$$\boxed{E = 0}$$

In Electrostatics

3 Main things, charge, electric field, pot^n.



Ques :- The pot^n of a charge distribution is given by

$$V(r, \theta, \phi) = \left(\frac{V_0}{2}\right)\left(3 - \frac{r^2}{R^2}\right) \quad , r < R$$

$$= \frac{V_0 R}{r} \quad , r > R$$

where V_0 & R are constants.

(a) Obtain the corresponding Electric field distribution.

(b) Obtain the charge distribution ρ.

(c) If R is the radius of sphere which is centred at the origin. Calculate the total charge enclosed by this sphere.

(a) $\vec{E} = -\vec{\nabla} V = -\frac{\partial V}{\partial r}\hat{r}$

$r < R$, $\vec{E} = -\vec{\nabla}\left[\frac{V_0}{2}\left(3 - \frac{r^2}{R^2}\right)\right]$

$$= -\frac{V_0}{2}\left(\cancel{3} - \frac{2r}{R^2}\right)\hat{r}$$

$$\boxed{\vec{E} = \frac{V_0 r}{R^2}\hat{r}}$$

$r > R$, $\vec{E} = -\frac{\partial V}{\partial r}\hat{r} = -\frac{\partial}{\partial r}\left(\frac{V_0 R}{r}\right)\hat{r}$

$$\vec{E} = -V_0 R\left(\frac{-1}{r^2}\right)\hat{r}$$

$$\boxed{\vec{E} = \frac{V_0 R}{r^2}\hat{r}}$$

(b) $\rho = ?$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow \rho = \epsilon_0(\vec{\nabla}\cdot\vec{E})$$

$r < R$, $\rho = \epsilon_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r)\right]$

$$= \epsilon_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{V_0 r}{R^2}\right)\right]$$

$$= \epsilon_0\left[\frac{1}{r^2}\frac{V_0}{R^2}3r^2\right]$$

$$\boxed{\rho = \frac{3V_0\epsilon_0}{R^2}}$$

$r > R$, $\rho = \epsilon_0\left[\cancel{\frac{1}{r}}\vec{\nabla}\cdot\left(V_0 R\frac{\hat{r}}{r^2}\right)\right]$

$$= \epsilon_0 V_0 R\, 4\pi \cancel{4}\, \delta^3(r)$$

~~for $r > R$~~, is not enclosing R. ~~So~~ $\delta^3(r) = 0$.

$$\therefore \boxed{\rho = 0}$$

(c)

$$q = \int \rho\, d\tau$$

$\{r < R,\}$ $q = \int_0^R \frac{3V_0\epsilon_0}{R^2} 4\pi r^2 dr$

$$= \frac{3V_0\epsilon_0}{R^2} 4\pi \frac{R^3}{3}$$

$$\boxed{q = 4\pi \epsilon_0 R V_0}$$

This is the total charge enclosed by this sphere.

Imp

Que 1- Electric potⁿ in some configuration is given by

$$V(r) = A \frac{e^{-\lambda r}}{r} \qquad A \text{ \& } \lambda \text{ are constants.}$$

find (i) ρ

(ii) $\vec{E}$

(iii) Total q in the sphere of radius R centred at the origin.

(i) $\vec{E} = -\vec{\nabla} V = -\frac{\partial V}{\partial r}\hat{r}$

$$\vec{E} = -\frac{\partial}{\partial r}\left[A\frac{e^{-\lambda r}}{r}\right]\hat{r}$$

$$= -A\left[\frac{1}{r}(-\lambda)e^{-\lambda r} + e^{-\lambda r}\left(\frac{-1}{r^2}\right)\right]\hat{r}$$

$$= A\left[\frac{\lambda e^{-\lambda r}}{r} + \frac{e^{-\lambda r}}{r^2}\right]\hat{r}$$

$$\vec{E} = A\left[\frac{\lambda}{r} + \frac{1}{r^2}\right]e^{-\lambda r}\hat{r} = A e^{-\lambda r}[\lambda r + 1]\frac{\hat{r}}{r^2}$$

(ii) $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$

$$\rho = \epsilon_0\left[\vec{\nabla}\cdot\vec{E}\right] = \epsilon_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A\left\{\frac{\lambda}{r} - \frac{1}{r^2}\right\}e^{-\lambda r}\right)\right]$$

$$= \cancel{A}\,\epsilon_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left[(\lambda r - 1)e^{-\lambda r}\right]\right]$$

$$= A\epsilon_0\left[\frac{1}{r^2}\left\{\lambda e^{-\lambda r} + (\lambda r - 1)e^{-\lambda r}(-\lambda)\right\}\right]$$

$$= A\epsilon_0\left[\frac{1}{r^2}\left\{\lambda e^{-\lambda r} - \lambda^2 r e^{-\lambda r} + \lambda e^{-\lambda r}\right\}\right]$$

$$= A\epsilon_0\left[\frac{1}{r^2}\left(2\lambda e^{-\lambda r} - \lambda^2 r e^{-\lambda r}\right)\right]$$

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$$\rho = \frac{A \epsilon_0}{r^2} \lambda e^{-\lambda r}(2 - \lambda r)$$

$$\rho = \epsilon_0 [\vec{\nabla} \cdot \vec{E}] = \epsilon_0 \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 A \frac{\lambda}{r} e^{-\lambda r} \right) \right]$$

$$+ \epsilon_0 \left[A \, \vec{\nabla} \cdot \left(e^{-\lambda r} \frac{\hat{r}}{r^2} \right) \right]$$

$$= \epsilon_0 \left[\frac{1}{r^2} A \lambda \left(r(-\lambda) e^{-\lambda r} + e^{-\lambda r} \right) \right]$$

$$+ \epsilon_0 A \left[\frac{\hat{r}}{r^2} (-\lambda) e^{-\lambda r} + e^{-\lambda r} 4\pi \delta^3(r) \right]$$

$$\{ \vec{\nabla} (\lambda \cdot \vec{A}) = \vec{A} \, \vec{\nabla} \cdot \lambda + \lambda \vec{\nabla} \cdot \vec{A} \}$$

$$= \epsilon_0 \left[\frac{A\lambda}{r^2} \{ (-\lambda r + 1) e^{-\lambda r} \} \right] + \epsilon_0 A \left[\frac{-\lambda e^{-\lambda r}}{r^2} + 4\pi \delta^3(r) \right]$$

$$= \frac{\epsilon_0 A \lambda}{r^2} e^{-\lambda r} - \frac{\epsilon_0 A \lambda^2}{r} e^{-\lambda r} - \frac{\epsilon_0 A \lambda e^{-\lambda r}}{r^2} + \epsilon_0 A \, 4\pi \delta^3(r)$$

$$\rho = -\frac{\epsilon_0 A \lambda^2}{r} e^{-\lambda r} + 4\pi \delta^3(r) \epsilon_0 A$$

$$\rho = \epsilon_0 A \left[\frac{-\lambda^2 e^{-\lambda r}}{r} + 4\pi \delta^3(r) \right]$$

<u>OR</u>

$$\rho = \epsilon_0 \left[\vec{\nabla} \cdot \{ \underbrace{A e^{-\lambda r} (\lambda r + 1)}_{\mathrm{I}} \cdot \underbrace{\frac{\hat{r}}{r^2}}_{\mathrm{II}} \} \right]$$

$$= \epsilon_0 A \left[\frac{\hat{r}}{r^2} \cdot \vec{\nabla} \left(e^{-\lambda r} (1 + \lambda r) \right) + e^{-\lambda r} (1 + \lambda r) \vec{\nabla} \cdot \frac{\hat{r}}{r^2} \right]$$

$$= \epsilon_0 A \left[\frac{\hat{r}}{r^2} \left(-\lambda e^{-\lambda r} + \lambda e^{-\lambda r} - \lambda^2 r e^{-\lambda r} \right) \cdot \hat{r} + e^{-\lambda r} \left(4\pi \delta^3(r) (1 + \lambda r) \right) \right]$$

$$\rho = \epsilon_0 A \left[\frac{-\lambda^2}{r} e^{-\lambda r} + 4\pi \delta^3(r) \right]$$

(Using $f(r) \cdot 4\pi \delta^3(r) = f(0) 4\pi \delta^3(r)$

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$$q = \int \rho \, d\tau \; \cancel{+\lambda \epsilon_0}$$

$$= -\epsilon_0 A \lambda^2 \int_0^R \frac{e^{-\lambda r}}{r} 4\pi r^2 dr + 4\pi \epsilon_0 A \int_V \underbrace{\delta^3(r)\, dr}_{1}$$

$$= -4\pi \epsilon_0 A \lambda^2 \int_0^R e^{-\lambda r} \cdot r\, dr + 4\pi \epsilon_0 A$$

$$= -4\pi \epsilon_0 A \lambda^2 \left[r \cdot \frac{e^{-\lambda r}}{-\lambda} - \frac{e^{-\lambda r}}{+\lambda^2} \right]_0^R + 4\pi \epsilon_0 A$$

$$= -4\pi \epsilon_0 A \lambda^2 \left[\frac{r e^{-\lambda r}}{-\lambda} - \frac{e^{-\lambda r}}{\lambda^2} \right]_0^R + 4\pi \epsilon_0 A$$

$$= -4\pi \epsilon_0 A \lambda^2 \left[\frac{R e^{-\lambda R}}{-\lambda} - \frac{e^{-\lambda R}}{\lambda^2} + \cancel{\frac{1}{\lambda^2}} \right] \cancel{+ 4\pi \epsilon_0 A}$$

$$= 4\pi \epsilon_0 A \left[e^{-\lambda R} + R\lambda e^{-\lambda R} \right]$$

$$\boxed{q = 4\pi \epsilon_0 A [1 + R\lambda] e^{-\lambda R}}$$

$\Rightarrow$ $\nabla^2 V = -\dfrac{\rho}{\epsilon_0}$

$\rho = ?$, $q = ?$

$$\nabla^2 V = \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) \right)$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial}{\partial r} \left(A \frac{e^{-\lambda r}}{r} \right) \right]$$

$$= \frac{A}{r^2} \frac{\partial}{\partial r} \left[r^2 \left(\frac{-\lambda e^{-\lambda r}}{r} - \frac{e^{-\lambda r}}{r^2} \right) \right]$$

$$= -\frac{A}{r^2} \frac{\partial}{\partial r} \left[(\lambda r + 1) e^{-\lambda r} \right]$$

$$= -\frac{A}{r^2} \left[(\lambda r + 1)(-\lambda) e^{-\lambda r} + e^{-\lambda r} (\lambda) \right]$$

$$= \frac{A}{r^2} \left[\lambda^2 r e^{-\lambda r} - \cancel{\lambda e^{-\lambda r}} + \cancel{\lambda e^{-\lambda r}} \right]$$

$$= \frac{A}{r} \lambda^2 e^{-\lambda r}$$

$$\rho = -\epsilon_0 \frac{A}{r} \lambda^2 e^{-\lambda r}$$

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$$q = \int \rho \, d\tau = \int_0^R -\frac{\epsilon_0 A \lambda^2 e^{-\lambda r}}{r} 4\pi r^2 dr$$

$$= -\epsilon_0 A \lambda^2 4\pi \int_0^R r e^{-\lambda r} dr$$

$$= -4\pi \epsilon_0 A \lambda^2 \left[-r \frac{e^{-\lambda r}}{\lambda} + \frac{e^{-\lambda r}}{-\lambda} \right]_0^R$$

$$= -4\pi \epsilon_0 A \lambda^2 \left[-\lambda R e^{-\lambda R} - e^{-\lambda R} + 0 + 1 \right]$$

$$= -4\pi \epsilon_0 A \lambda^2 \left[-e^{-\lambda R} - \lambda R e^{-\lambda R} + 1 \right]$$

$$= 4\pi \epsilon_0 A \lambda^2 \left[1 - e^{-\lambda r} (1 - \lambda R) \right]$$

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$$= 4\pi \epsilon_0 A \lambda \left[r e^{-\lambda r} + e^{-\lambda r} \right]_0^R$$

$$= 4\pi \epsilon_0 A \lambda \left[R e^{-\lambda R} + e^{-\lambda R} - 1 \right]$$

$$q = 4\pi \epsilon_0 A \lambda \left[e^{-\lambda R} (R+1) - 1 \right]$$

Qu :- Yukava Potⁿ is given by $V = \frac{-q}{r} e^{-\alpha t} \left(1 + \frac{\alpha r}{2}\right)$

Find (i) $\vec{E}$ (ii) ρ (iii) Total q in a sphere of radius R.

(i) $\vec{E} = -\vec{\nabla} V = -\frac{\partial V}{\partial r} \hat{r}$

$$\vec{E} = \frac{-\partial}{\partial r} \left[\frac{-q}{r} e^{-\alpha t} \left(1 + \frac{\alpha r}{2}\right) \right] \hat{r} = q e^{-\alpha t} \frac{\partial}{\partial r} \left[\frac{1}{r} \left(1 + \frac{\alpha r}{2}\right) \right] \hat{r}$$

$$= q e^{-\alpha t} \frac{\partial}{\partial r} \left[\frac{1}{r} + \frac{\alpha}{2} \right] \hat{r}$$

$$= q e^{-\alpha t} \left[\frac{-1}{r^2} \right] \hat{r} = -\frac{q e^{-\alpha t}}{r^2} \hat{r}$$

$$\boxed{\vec{E} = -\frac{q e^{-\alpha t}}{r^2} \hat{r}}$$

(ii) $\rho = ?$ $\quad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

$$\rho = \epsilon_0 \left[-q e^{-\alpha t} \left[\vec{\nabla} \cdot \frac{\hat{r}}{r^2} \right] \right] = -q \epsilon_0 e^{-\alpha t} 4\pi \delta^3(r)$$

$$\boxed{\rho = -4\pi \epsilon_0 q e^{-\alpha t} \delta^3(r)}$$

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$$Q = \int \rho\, d\tau = \int_0^R \cancel{-4\pi\epsilon_0 q e^{-\alpha t}\, \delta^3(r)\, 4\pi r^2 dr}$$

$$\cancel{= (4\pi)^2 \epsilon_0}$$

$$= -4\pi\epsilon_0 q e^{-\alpha t} \int_V \delta^3(r)\, dr$$

{sphere is centred at origin so $\int_V \delta^3(r)\, d\tau = 1$}

$$\boxed{Q = -4\pi\epsilon_0 q e^{-\alpha t}}$$

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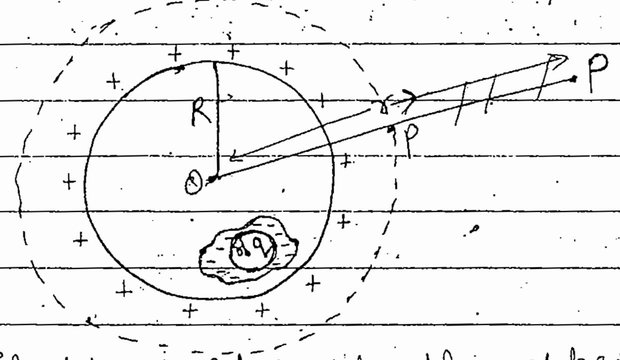
Q.1:- An uncharged spherical conductor centred at the origin has a cavity of some arbitrary shape. Somewhere within the cavity charge is q. Then find electric field inside & outside the sphere.



Empty space inside the sphere is cavity.

Electric field inside the sphere

$$\boxed{\vec{E}_{in} = 0}$$

Electric field outside the sphere

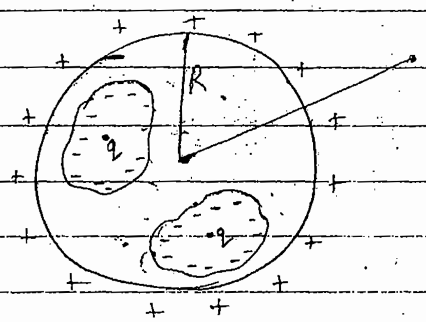
$$\boxed{\vec{E}_{out} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}}$$

Electric field inside & outside independent on shape on the cavity but dependent on charge of the cavity.

Electric field within the cavity, (it is not 0 bcoz it enclosing the charge q)

$$\boxed{E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}}$$

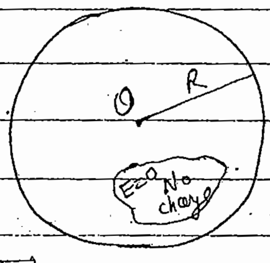
⟹ If we have more than one cavity.



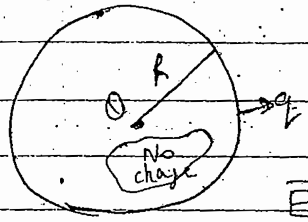
Note:- Charges inside the different cavities are independent of each other.

⟹ If we have a conducting sphere of radius R. We have a cavity which is empty itself, Electric field inside the cavity is ZERO. (wheather the conductor is charged or not)

If conductor is charged.



$\vec{E} = 0$



still $\vec{E} = 0$.

Q.2 - A metal sphere of radius R carrying a small charge q is surrounded by a thick cocentric metal shell inner radius 'a' & outer radius 'b'. The shell carries no net charge.

(a) Find the charge densities at a, b & R

(b) Find the potⁿ at centre using ∞ as a reference point.

(c) Find the electric field for

(i) $r < R$

(ii) $R < r < a$

(iii) $a < r < b$

(iv) $r > b$

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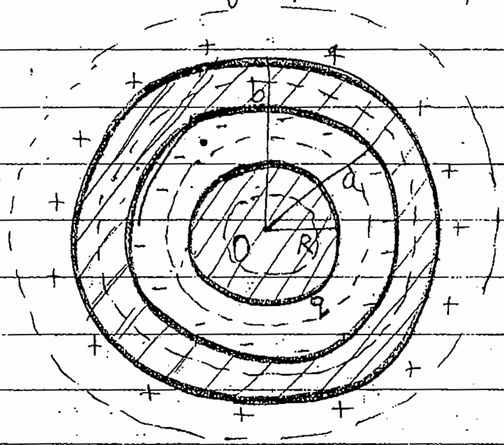
(d) Now if the outer surface is touched to the ground how the answers of part (a), (b), (c) will be change?



(a) charge density

$$\sigma = \frac{\text{charge}}{\text{Area}}$$

$q = \sigma A$

$\sigma = \frac{q}{A}$

$$\sigma_R = \frac{q}{4\pi R^2}$$

$$\sigma_a = \frac{-q}{4\pi a^2}$$

$$\sigma_b = \frac{q}{4\pi b^2}$$

(b) $V_{centre} = \frac{q}{4\pi\epsilon_0 R} - \frac{q}{4\pi\epsilon_0 a} + \frac{q}{4\pi\epsilon_0 b}$

$$V_{centre} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R} - \frac{1}{a} + \frac{1}{b}\right]$$

(c) $\vec{E} = ?$

for $r < R$, $\boxed{E = 0}$ (No charge)

for $R < r < a$ $\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}}$

$r < b$ $\boxed{\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}}$

$a < r < b$ $\boxed{\vec{E} = 0}$

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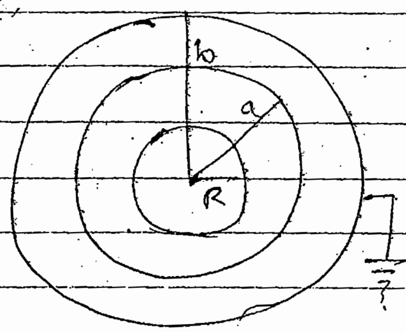
(d) Now outer surface is grounded.

On σ_a & σ_R → No effect

$\sigma_b = 0$



$$V_{centre} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R} - \frac{1}{a}\right]$$

$\vec{E}$ in region $r > b$

$$\vec{E} = 0$$

In others region → No effect.

Q3:- Two spherical cavities of radii a & b are hollowed out from the interior of a neutral conducting sphere of radius R at the centre of each cavity point charge q_a & q_b ~~having radii a & b respectively~~ are placed. ~~q_a charge~~ Find

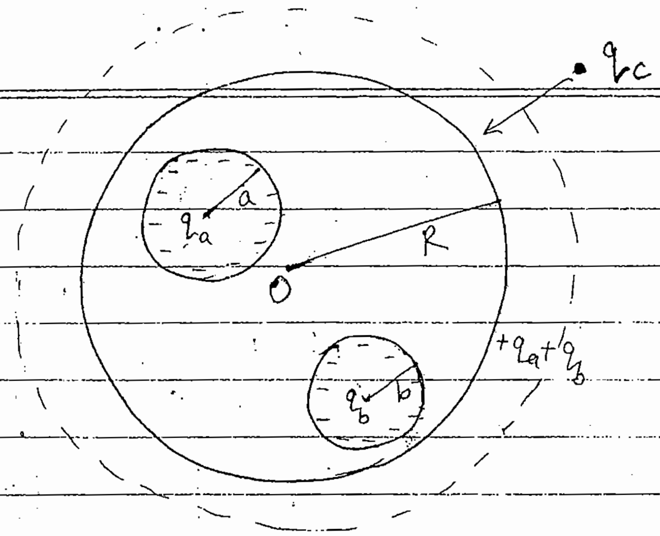
(a) The surface charge densities σ_a, σ_b & σ_R.
(b) Electric field inside & outside the conductor
(c) " " " the cavities.
(d) force ~~on~~ b/w charges q_a & q_b.
(e) Which of the answers (a), (b), (c) & (d) will change if a III charge q_c were brought near the conductor.

(a) $$\sigma_a = \frac{-q_a}{4\pi a^2}$$

$$\sigma_b = \frac{-q_b}{4\pi b^2}$$

$$\sigma_R = \frac{q_a + q_b}{4\pi R^2}$$

(b) $$\boxed{\vec{E}_{in} = 0}$$



$$\vec{E}_{out} = \frac{q_a+q_b}{4\pi\epsilon_0 r^2}\hat{r}$$

(c) Electric field inside the cavities

$$\frac{q_a}{4\pi\epsilon_0 r^2}\hat{r} \quad , \quad \frac{q_b}{4\pi\epsilon_0 r^2}\hat{r}$$

(d) $\boxed{F=0}$

$[F_a = 0 \quad , F_b = 0]$

(e) $\sigma_a, \sigma_b \rightarrow$ No change

σ_R will change. Now charge will be Non-uniform.

$E_{in} = 0$

$E_{out} = \frac{q_a+q_b}{4\pi\epsilon_0 r^2}\hat{r}$ will change

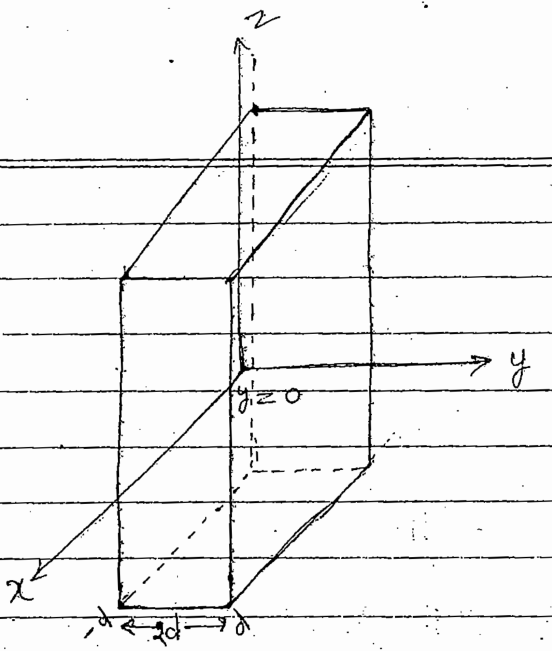
$\vec{E}$ inside the cavity $\rightarrow$ No change

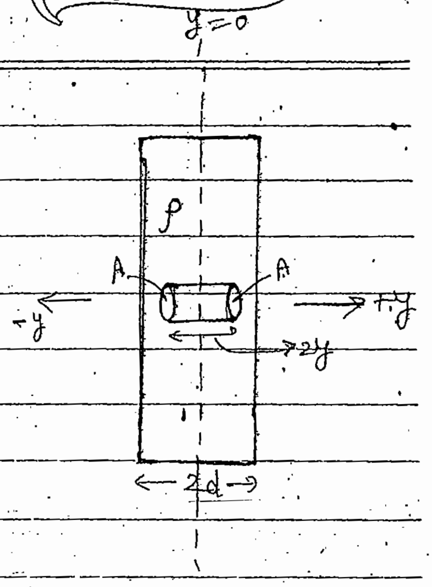
force " " $\rightarrow$ No change.

Q.4 :- An infinite plane slab of thickness 2d carries a uniform charge density ρ. Find the electric field ~~in~~ as a fun^n of y where y=0 at the centre. Plot $\vec{E}$ v/s y. ~~Only E~~ Calling E +ve when it points in +y dir^n & -ve when it points in -y dir^n.

conducting → $E = \frac{\sigma}{\epsilon_0}$
Non- " → $E = \frac{\sigma}{2\epsilon_0}$

$y=-d \quad y=+d$





dir^n of $\vec{E}$ will be in $+y$ or $-y$ dir^n. We can also take a Gaussian pillbox instead of cylender as a Gaussian surface.

Charge enclosed in the cylender

$$q_{enc} = \rho \cdot A \cdot 2y$$

$$\left\{ \begin{array}{l} vol = \pi R^2 l \\ \quad = A \cdot l \\ \rho = q/vol. \end{array} \right.$$

The flux passing through both cross section surface is $\phi = 2E \cdot A$

$$2EA = \frac{q}{\epsilon_0}$$

$$2EA = \frac{\rho A\, 2y}{\epsilon_0}$$

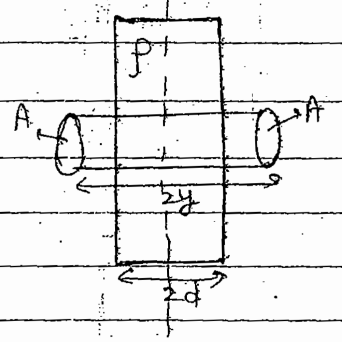
$$\boxed{\begin{array}{ll} \vec{E}_{in} = \frac{\rho y}{\epsilon_0}\hat{y} & 0<y<d \\ \vec{E}_{in} = \frac{\rho y}{\epsilon_0}(-\hat{y}) & -d<y<0 \end{array}}$$

For Outside :-

$$q_{enc} = \rho \cdot A \cdot 2d$$

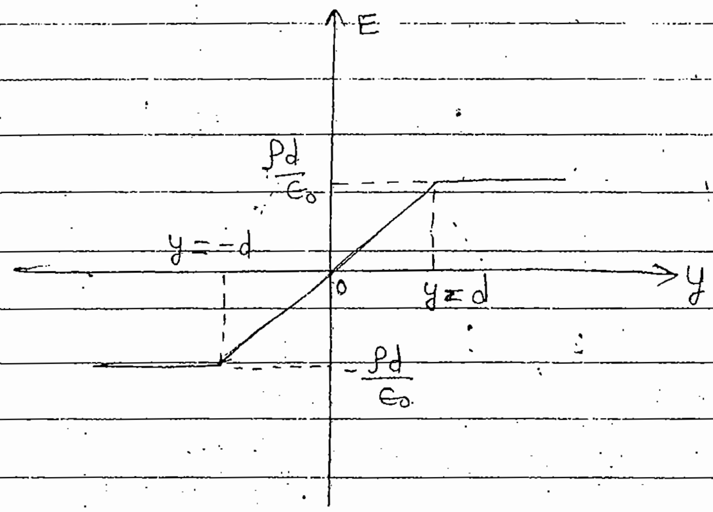
$$flux = 2E \cdot A$$

$$2E \cdot A = \frac{\rho A 2d}{\epsilon_0}$$



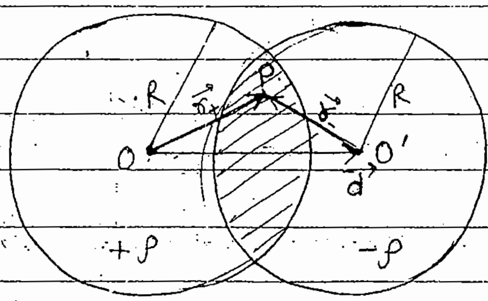
$$\boxed{\begin{array}{ll} \vec{E}_{out} = \frac{\rho d}{\epsilon_0}\hat{y} & y>d \\ \vec{E}_{out} = \frac{\rho d}{\epsilon_0}(-\hat{y}) & y<-d \end{array}}$$

Behaviour of $\vec{E}$:- Plot $\vec{E}$ v/s y.



⇒ Outside of a thin conducting sheet $\vec{E}$ is constant.

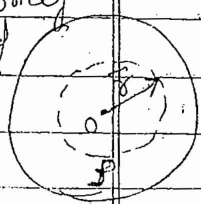
Q.51 Two spheres each of radius R carrying the uniform charge densities $+\rho$ & $-\rho$ respectively are placed so that they partially overlap. Take a vector from the +ve centre to -ve centre. $\vec{d}$. Show that the field in the region of overlap is constant & find its value.



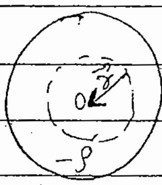
$\vec{OO'} = \vec{d}$
$\vec{OP} = \vec{r}_+$
$\vec{O'P} = \vec{r}_-$

$\vec{E}$ follow the superposition principle.

uniformly charged sphere



$$\vec{E} = \frac{\rho r}{3\epsilon_0}\hat{r} = \frac{\rho\vec{r}}{3\epsilon_0}$$



$$\vec{E} = \frac{\rho r}{3\epsilon_0}(-\hat{r})$$

$q_{enc} = \int \rho \, d\tau$

$= \int \rho \, 4\pi r^2 \, dr = \rho \, 4\pi \frac{r^3}{3}$

$E \cdot 4\pi r^2 = \frac{q}{\epsilon} = \frac{\rho \, 4\pi r^3}{3\epsilon}$

$\boxed{E = \frac{\rho r}{3\epsilon} \hat{r}}$

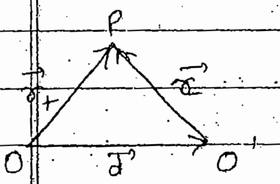
Electric field due to the $+\rho$ & $-\rho$ will be

$$\vec{E}_+ = \frac{\rho \vec{r}_+}{3\epsilon_0} \quad \text{——— (1)}$$

$$\vec{E}_- = \frac{\rho(-\vec{r}_-)}{3\epsilon_0} \quad \text{——— (2)}$$

Total Electric field $\vec{E} = \vec{E}_+ + \vec{E}_-$

$$\vec{E} = \frac{\rho}{3\epsilon_0}(\vec{r}_+ - \vec{r}_-)$$



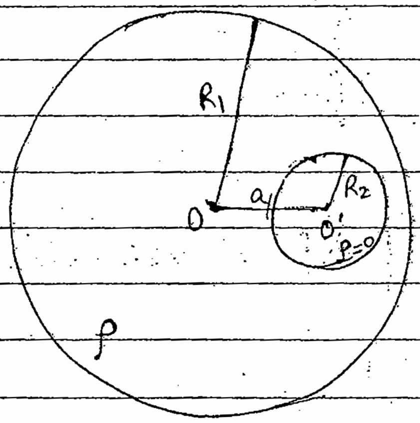
$$\boxed{\vec{E} = \frac{\rho \vec{d}}{3\epsilon_0}}$$

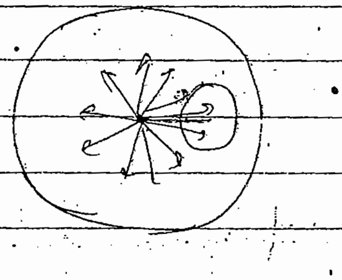
$$\left\{ \begin{array}{l} \vec{r}_+ = \vec{d} + \vec{r}_- \\ \vec{r}_+ - \vec{r}_- = \vec{d} \end{array} \right\}$$

$d \rightarrow$ constant

so $\vec{E}$ is constant

Q.6 :- A sphere of radius R_1 has a charge density ρ uniform within its volume except for a small spherical hollow region of radius R_2 located at a distance a from the centre. Find the electric field inside the hollow sphere.





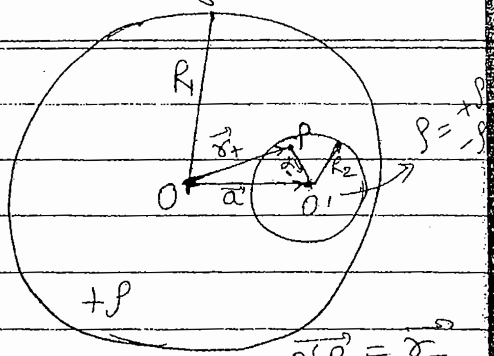
In case of conducting sphere $q = 0$ inside the sphere so $\vec{E} = 0$ i.e. no electric field lines. But here uniformly charged sphere having charge uniformly dis^{ted} so field lines radially outward & will pass through the cavity also. so In cavity $\vec{E} \rightarrow$ Non. zero.

In Cavity $\rho = 0$

let $\rho = \rho_+ + \rho_-$

$\rho = +\rho - \rho$

Electric field inside the cavity = ?



$\vec{O'P} = \vec{r}_-$

$\vec{OP} = \vec{r}_+$

$\vec{OO'} = \vec{a}$

$$\vec{E}_+ = \frac{\rho \vec{r}_+}{3\epsilon_0} \quad \text{---(1)}$$

$$\vec{E}_- = \frac{\rho(-\vec{r}_-)}{3\epsilon_0} \quad \text{---(2)}$$

Total $\vec{E} = \vec{E}_+ + \vec{E}_-$

$$\vec{E} = \frac{\rho}{3\epsilon_0}(\vec{r}_+ - \vec{r}_-)$$

$$\boxed{\vec{E} = \frac{\rho \vec{a}}{3\epsilon_0}}$$ depande on the distance b/w centre of the sphere & centre of cavity

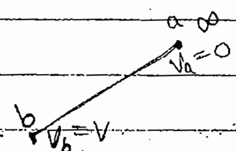
$\vec{a} \to$ Constant. So Electric field inside the cavity is constant.

Note :- If centre of sphere & centre of cavity Conside then $\vec{E} = 0$

- Energy :- Work done to move a point charge

Work done = qV

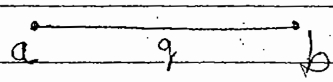
$$\boxed{W = qV}$$ (finite)



If we want to take a point charge from ∞ to a a point ~~will~~ then work done to move a charge will store in terms of 'potential Energy'.

If we want to move a charge from point a to b then

$$\boxed{W = q[V_b - V_a]}$$



W can be written as

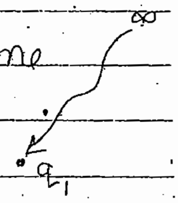
$$\boxed{W = -q\int_{\infty}^{r} \vec{E}\cdot d\vec{r} = -q\int_{a}^{b} \vec{E}\cdot d\vec{r}}$$

Potential will be $V = \frac{W}{q}$

$$\boxed{V = -\int_{\infty}^{r} \vec{E}\cdot d\vec{r} = -\int_{a}^{b} \vec{E}\cdot d\vec{r}}$$

Work done or Potential Energy to form a system of charges :-

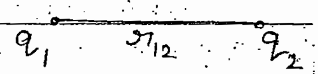
If we have q_1, q_2, q_3 → 3 charges.



To ~~m~~ take charge q_1 from ∞, No work is done bcoz at that point there was no charge before so $\vec{E}$ was zero. $\boxed{W_1 = 0}$ —— (1)

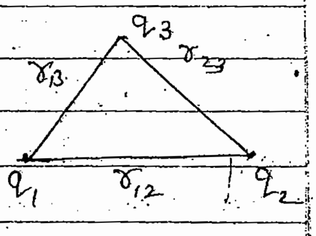
Now take q_2 to that point from ∞ then work done,

$$W_2 = q_2 V_1$$



$$W_2 = \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}} \quad \text{—— (2)}$$

Now take q_3 from ∞ to the point. So there must be some work done.



$$W_3 = q_3 V_1 + q_3 V_2$$

$$W_3 = \frac{q_1 q_3}{4\pi\epsilon_0 r_{13}} + \frac{q_2 q_3}{4\pi\epsilon_0 r_{23}} \quad \text{—— (3)}$$

Total Work done to assemble the system,

$$W = W_1 + W_2 + W_3$$

$$\boxed{W = \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}} + \frac{q_1 q_3}{4\pi\epsilon_0 r_{13}} + \frac{q_2 q_3}{4\pi\epsilon_0 r_{23}}}$$

The work done to assemble the system will store as the potⁿ energy of the system. $\boxed{W = E}$

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Generalized for n-charges :- Suppose we have n charges then Q.2

$$\boxed{W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{q_i q_j}{r_{ij}}}$$, where $j > i$

The correct form is

$$\boxed{W = \frac{1}{2} \times \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{q_i q_j}{r_{ij}}} \quad i \neq j$$

$$\Rightarrow \boxed{W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{q_i q_j}{r_{ij}}} \quad i \neq j$$

This formula is applicable only for point charges. Not applicable for continous charges.
from this formula we can find Energy of the system directly.

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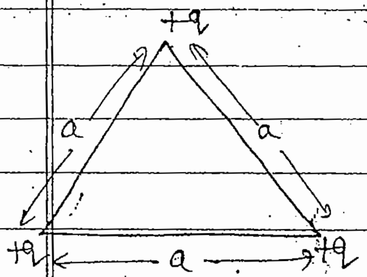
Que:- Calculate the energy of this configuration.

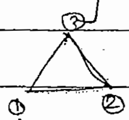


Energy of the system of 3 charges will be

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{q_i q_j}{r_{ij}}$$

$$W = \frac{1}{8\pi\epsilon_0}\left[\frac{q_1q_2}{a} + \frac{q_2q_3}{a} + \frac{q_3q_1}{a} + \frac{q_2q_1}{a} + \frac{q_3q_2}{a} + \frac{q_1q_3}{a}\right]$$

$$\cancel{W = \frac{1}{8\pi\epsilon_0}\left[\frac{3qq}{a}\right]}$$



$$W = \frac{1}{8\pi\epsilon_0}\left[\frac{6q^2}{a}\right]$$

$$\boxed{W = \frac{3q^2}{4\pi\epsilon_0 a}}$$

$W_1 = 0$

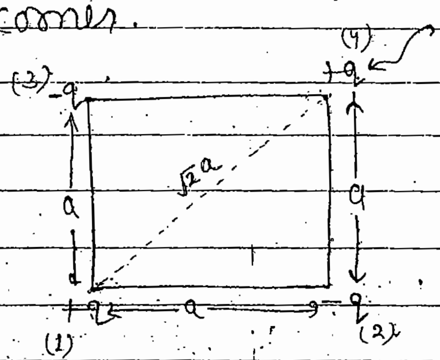
$W_2 = qV_1 = \frac{q^2}{4\pi\epsilon_0 a}$

$W_3 = qV_1 + qV_2 = \frac{q^2}{4\pi\epsilon_0 a} + \frac{q^2}{4\pi\epsilon_0 a}$

$W = W_1 + W_2 + W_3$

$$\boxed{W = \frac{3q^2}{4\pi\epsilon_0 a}}$$

Q.2 :- 3 charges are situated at the corners of a square of sides a. How much work done required to bring a 4th charge +q to place at the 4th corner.



$$W_4 = \frac{1}{4\pi\epsilon_0}\frac{q^2}{\sqrt{2}a} - \frac{1}{4\pi\epsilon_0}\frac{q^2}{a} - \frac{1}{4\pi\epsilon_0}\frac{q^2}{a} + 0$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q^2}{a}\left[\frac{1}{\sqrt{2}} - 2\right]$$

$$\boxed{W = \frac{q^2}{4\pi\epsilon_0 a}\left[\frac{1-2\sqrt{2}}{\sqrt{2}}\right]}$$

Work done required to assemble this system,

$$W = \frac{1}{8\pi\epsilon_0}\left[\frac{q_1q_2}{a} + \frac{q_1q_3}{a} + \frac{q_1q_4}{\sqrt{2}a} + \frac{q_2q_1}{a} + \frac{q_2q_3}{\sqrt{2}a} + \frac{q_2q_4}{a} + \frac{q_3q_1}{a} + \frac{q_3q_2}{\sqrt{2}a} + \frac{q_3q_4}{a} + \frac{q_4q_1}{\sqrt{2}a} + \frac{q_4q_2}{a} + \frac{q_4q_3}{a}\right]$$

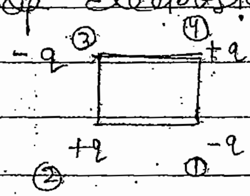
$$W = \frac{1}{8\pi\epsilon_0}\left[-\frac{q^2}{a} - \frac{q^2}{a} + \frac{q^2}{\sqrt{2}a} - \frac{q^2}{a} + \frac{q^2}{\sqrt{2}a} - \frac{q^2}{a} - \frac{q^2}{a} + \frac{q}{\sqrt{2}a} + -\frac{q^2}{a} + \frac{q^2}{\sqrt{2}a} - \frac{q^2}{a} - \frac{q^2}{a}\right]$$

$$= \frac{1}{8\pi\epsilon_0}\left[-\frac{8q^2}{a} + \frac{4q^2}{\sqrt{2}a}\right] = \frac{q^2}{8\pi\epsilon_0 a}\left[-8 + \frac{4}{\sqrt{2}}\right]$$

$$= \frac{q^2}{8\pi\epsilon_0 a}\left[\frac{-8+2\sqrt{2}\times 2}{\sqrt{2}\times 2}\right] = \frac{4q^2}{8\pi\epsilon_0 a}\left[-2 + \frac{1}{\sqrt{2}}\right]$$

$$\boxed{W = \frac{2q^2}{4\pi\epsilon_0 a}\left[-2 + \frac{1}{\sqrt{2}}\right]}$$

This work done will store in terms of Electrostatic energy.



$W_1 = 0$

$W_2 = +qV_1$

$W_3 = -q(V_1+V_2) =$

$W_4 =$

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Energy of Continous Charges :-

$$\boxed{W = \frac{1}{2}\int_V \rho V\, d\tau = \frac{1}{2}\int_S \sigma V\, da = \frac{1}{2}\int_C \lambda V dl}$$

$V \to$ potential.

Another form -

$$\boxed{W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 d\tau}$$

Find the Energy of a point charge q :-

it means work done required to make a point charge.

Electric field on a point charge

$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$$

So $$W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 d\tau$$

$$= \frac{\epsilon_0}{2}\int_0^\infty \frac{q^2}{16\pi^2\epsilon_0^2 r^4} 4\pi r^2 dr$$

$$= \frac{\epsilon_0}{2}\frac{q^2}{4\pi\epsilon_0^2}\left[\frac{r^{-1}}{-1}\right]_0^\infty = \frac{q^2}{8\pi\epsilon_0}\left[\frac{-1}{r}\right]_0^\infty$$

$$W = \frac{-q^2}{8\pi\epsilon_0}\left[\frac{1}{\infty} - \frac{1}{0}\right] = \infty$$

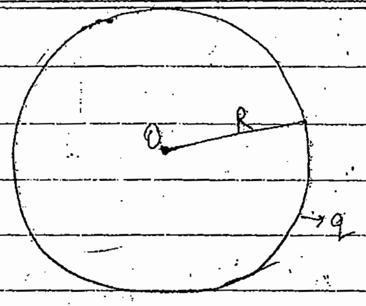
i.e. Work done to make a point charge is ∞. It means we can not make a point charge - They are found in the nature as it is.

Que :- Find the energy of a conducting sphere having radius R & total charge q.

Inside $\vec{E}_{in} = 0$ & $\vec{E}_{out} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$

$$W = \frac{\epsilon_0}{2}\left[\int_0^R \underset{0}{E_{in}^2}\, d\tau + \int_R^\infty E_{out}^2 d\tau\right]$$

$$W = \frac{\epsilon_0}{2}\int_R^{\infty} \frac{q^2}{(4\pi\epsilon_0)^2 r^4} 4\pi r^2 dr$$



$$= \frac{\epsilon_0}{2}\frac{q^2}{4\pi\epsilon_0^2}\left[-\frac{1}{r}\right]_R^{\infty}$$

$$= \frac{q^2}{8\pi\epsilon_0}\left[-\frac{1}{\infty}+\frac{1}{R}\right]$$

$$W = \frac{q^2}{8\pi\epsilon_0 R} \Rightarrow \boxed{U = \frac{q^2}{8\pi\epsilon_0 R}}$$

energy

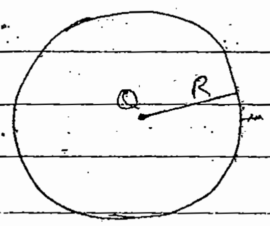
II method

$$U = \frac{1}{2}\int_S \sigma V da = \frac{1}{2}\oint \frac{q}{4\pi\epsilon_0 R} q$$

$$\boxed{U = \frac{q^2}{8\pi\epsilon_0 R}}$$

Que :- Find the energy of a uniformly charged sphere of radius R & total charge q.

$$E_{in} = \frac{qr}{4\pi\epsilon_0 R^3}\hat{r}, \quad E_{out} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$$



$$W = \frac{\epsilon_0}{2}\left[\int_0^R E_{in}^2 d\tau + \int_R^{\infty} E_{out}^2 d\tau\right]$$

$$W = \frac{\epsilon_0}{2}\left[\int_0^R \frac{q^2 r^2}{16\pi^2\epsilon_0^2 R^6} 4\pi r^2 dr\right] + \frac{q^2}{8\pi\epsilon_0 R}$$

$$= \frac{\epsilon_0}{2}\frac{q^2}{4\pi\epsilon_0^2 R^6}\int_0^R r^4 dr + \frac{q^2}{8\pi\epsilon_0 R}$$

$$= \frac{q^2}{8\pi\epsilon_0 R^6}\left[\frac{r^5}{5}\right]_0^R + \frac{q^2}{8\pi\epsilon_0 R}$$

$$= \frac{q^2}{8\pi\epsilon_0 R^6}\left[\frac{R^5}{5}\right] + \frac{q^2}{8\pi\epsilon_0 R}$$

$$= \frac{q^2}{8\pi\epsilon_0}\frac{1}{5R} + \frac{q^2}{8\pi\epsilon_0 R} = \frac{q^2}{8\pi\epsilon_0 R}\left[\frac{1}{5}+1\right]$$

$$\boxed{U = \frac{6}{5}\frac{q^2}{8\pi\epsilon_0 R}}$$

Note:- Work done in the case of uniformly charged sphere is more as compare to conducting sphere. Hence Energy is More.

Problem:- Charge density inside a Non-conducting sphere is given by $\rho = Kr$. If R is the radius of the sphere. find the energy of this sphere.

$$E_{in} = \frac{Kr^2}{4\epsilon_0}\hat{r} \quad \& \quad E_{out} = \frac{KR^4}{4\epsilon_0 r^2}\hat{r}$$

$$W = \frac{\epsilon_0}{2}\left[\int_0^R \cancel{\frac{Kr^2}{4\epsilon_0}} E_{in}^2\, d\tau + \int_R^\infty E_{out}^2\, d\tau\right]$$

$$W = \frac{\epsilon_0}{2}\left[\int_0^R \frac{K^2 r^4}{16\epsilon_0^2} 4\pi r^2 dr + \int_R^\infty \frac{K^2 R^8}{16\epsilon_0^2 r^4} 4\pi r^2 dr\right]$$

$$= \frac{\epsilon_0}{2}\left[\frac{K^2\pi}{4\epsilon_0^2}\left[\frac{r^7}{7}\right]_0^R + \frac{K^2\pi R^8}{4\epsilon_0^2}\left[\frac{-1}{r}\right]_R^\infty\right]$$

$$= \frac{\epsilon_0}{2}\left[\frac{K^2\pi}{4\epsilon_0^2}\cdot\frac{R^7}{7} - \frac{K^2\pi R^8}{4\epsilon_0^2}\left[0 - \frac{1}{R}\right]\right]$$

$$= \frac{\epsilon_0}{2}\left[\frac{K^2\pi}{4\epsilon_0^2}\cdot\frac{R^7}{7} + \frac{K^2\pi R^7}{4\epsilon_0^2}\right]$$

$$= \frac{\epsilon_0}{2}\left[\frac{K^2\pi R^7}{4\epsilon_0^2}\left[\frac{1}{7} + 1\right]\right]$$

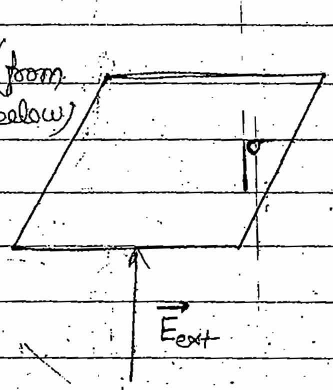
$$= \frac{\epsilon_0}{\cancel{2}}\left[\frac{K^2\pi R^7}{\cancel{4}\epsilon_0^2}\cdot\frac{\cancel{8}}{7}\right]$$

$$U = W = \frac{K^2\pi R^7}{7\epsilon_0}$$

<u>Electrostatic Boundry Conditions</u>:- Whenever electric field crosses the surface charge σ, it suffers a discontinuty & to find the amount of discontinuty, <u>Electrostatic boundry conditions</u> are used.

If we apply some external electric (from below) field on surface, then internal field will oppose it & suport the field which is above the surface.

The value of $\vec{E}$ will be different in above region & below region.



$\Longrightarrow$ These Boundry cond^n holds for surface only.

We'll find b.c. for 2 components of $\vec{E}$:

→ $E_\perp$ → normal comp.

→ $E_\parallel$ → tangential "

Consider a closed surface

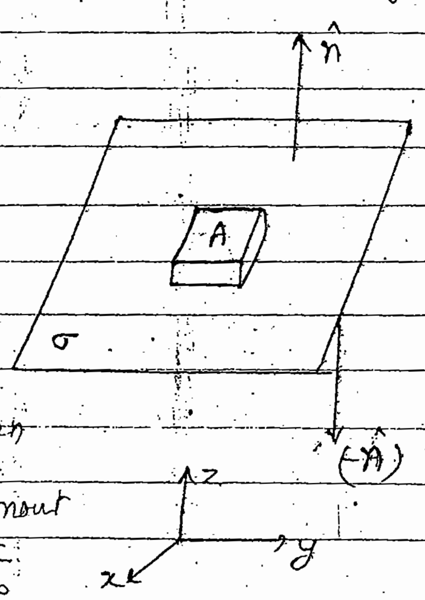
$$\oint \vec{E}\cdot d\vec{S} = \frac{q}{\epsilon_0}$$

Take a pillbox as a Gaussian surface (or cylender)

$$\Rightarrow E^{\perp}_{above} A - E^{\perp}_{below} A = \frac{\sigma A}{\epsilon_0}$$

$$\boxed{E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0}}$$

This is the discontinuty in normal Electric field.

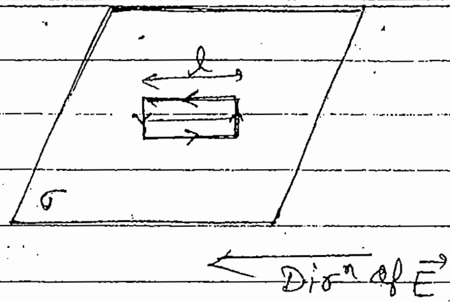


We apply E_0 Elec field from downward.
In below region, field will be less than E_0 by amount $\sigma/2\epsilon_0$ & In above region field will be more than E_0 by amout $\sigma/2\epsilon_0$. Total discontinuty $= \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0}$

$\oint E \cdot dl = 0$

$E^{\parallel}_{above}\, l - E^{\parallel}_{below}\, l = 0$

$\Rightarrow \boxed{E^{\parallel}_{above} = E^{\parallel}_{below}}$



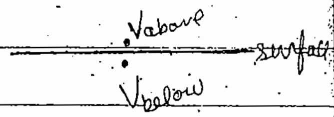
It means Tangential comp. of $\vec{E}$ ~~par~~ is continuous over the boundry.

$\Rightarrow$ Discontinuty occur in normal Dirⁿ.

Potential :- $V = -\int \vec{E} \cdot \vec{dl}$

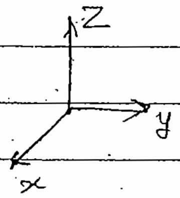
Near the boundry of surface

$\boxed{V_{above} = V_{below}}$



So Potⁿ is continuous near the boundry.

$\boxed{\vec{E} = -\vec{\nabla} V}$



If we go away from boundry then there will be some difference in V_{above} & V_{below}.

$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0}$

$-\frac{\partial V_{above}}{\partial z} + \frac{\partial V_{below}}{\partial z} = \frac{\sigma}{\epsilon_0}$

$$\boxed{\frac{\partial V_{above}}{\partial z} - \frac{\partial V_{below}}{\partial z} = -\frac{\sigma}{\epsilon_0}}$$

$$\boxed{\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\epsilon_0}}$$

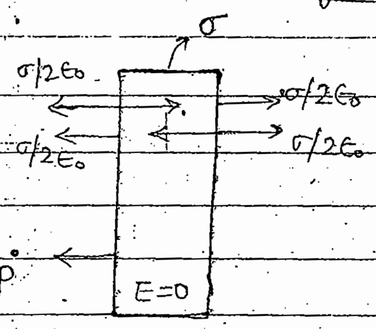
$n \rightarrow$ normal to the surface.

This hold for surface only.

There will be discontinuty in case of normal electric field.

Note :- If σ is given & we have to find V then use boundry cond^n & If ρ is given & find V then use Poission's eq^n $\nabla^2 V = -\rho/\epsilon_0$.

Force on a Conductor due to surface charge :-



Pressure on any point on surface = Elec. field due to all other charges $\times\ \sigma$ of that point.

$$F = qE_{in}$$

$$F = \sigma A E_{in}$$

$$P = |\vec{f}| = \frac{\vec{F}}{A} = \sigma E_{in} = \sigma \frac{\sigma}{2\epsilon_0}\hat{n}$$

$$\boxed{P = \frac{\sigma^2}{2\epsilon_0}\hat{n}}$$

Pressure $\boxed{P = \frac{\epsilon_0 E^2_{Tot}}{2}}$

Electrostatic pressure or force/unit area. Also called Electrostatic Energy density.

$$U = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 d\tau$$

$$\frac{U}{d\tau} = \frac{\epsilon_0}{2}\int E^2$$

$$\cancel{U} \neq \boxed{P = \frac{\epsilon_0}{2}\int E^2}$$

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Laplace Eqⁿ & Poisson's Eqⁿ :-

We have $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$ ——(1)

& $\vec{\nabla}\times\vec{E} = 0$ ——(2)

$\vec{E} = -\vec{\nabla} V$

(1) $\Rightarrow$ $\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}}$ → Possion's eqⁿ

If region is charge free then $\rho = 0$

$\therefore$ $\boxed{\vec{\nabla}^2 V = 0}$ → Laplace eqⁿ.

Note :- If $V \propto \frac{1}{r}$ ~~then~~ & we have to find ρ then Ist find $\vec{E}$ & then ρ But Except this in all other cases we use Poission's eqⁿ to find ρ if V given.

Uniqueness Theorem :- There are 2 theorems of uniqueness :-

(I) Potential (V) (II) Electric field (E)

(I) Theorem of Potⁿ :- If given potⁿ V satisfies the boundry condⁿ then this is the only solution to laplace eqⁿ i.e. This potⁿ is unique.

(II) Theorem of $\vec{E}$:- ~~If~~ in a volume V surrounded by conductors & containing a specified charge density ρ. The electric field is uniquely determine if total charge on each conductor is given.

Method Of Images :- There are 2 main problems

(1) Infinite grounded conducting plane & a point charge.

(2) Grounded conducting sphere & a point charge.

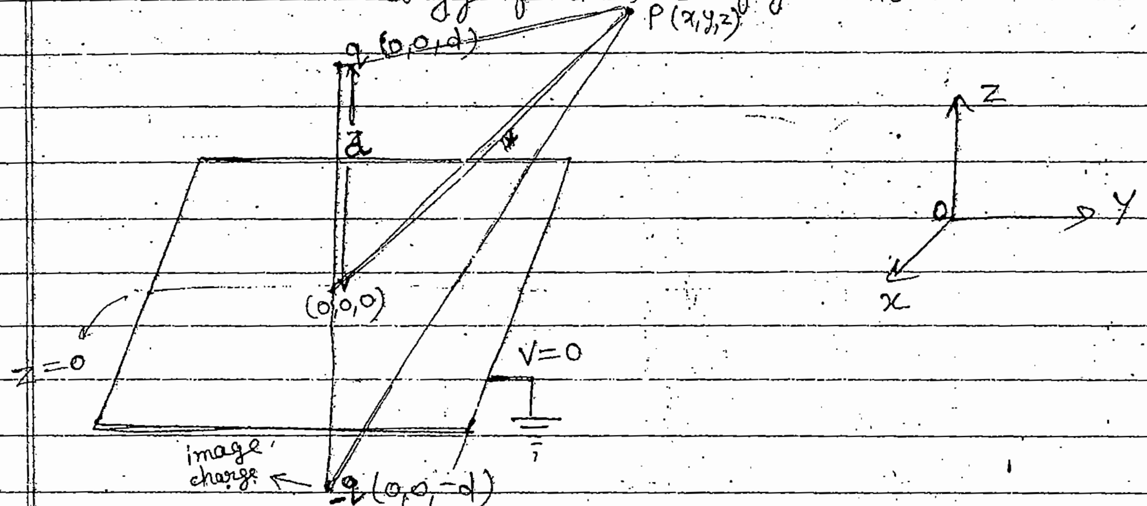
Que.1- A charge q is ~~had~~ at a distance d above an infinite grounded conducting plane. Find ϕ

(i) Potential.

(ii) Electric field in the region above the plane.

Also find force b/w plane & the charge.

" " Energy of this configuration.



B.C.s (i) $V=0$ at $z=0$

(ii) $V \to 0$ if $r^2 = x^2+y^2+z^2 >> d^2$

$$V = \frac{q}{4\pi\epsilon_0\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{4\pi\epsilon_0\sqrt{x^2+y^2+(z+d)^2}} \quad \text{--- (1)}$$

This is the pot^n of point charge q & image of point charge.

i.e. We replace the prob. of a point charge into the prob. of 2 point charges $\longrightarrow$ one q is at $(0,0,+d)$ & $-q$ at $(0,0,-d)$

$\boxed{V_{below} = 0}$

Electric field :- $\vec{E} = -\vec{\nabla} V$

$$= -\frac{\partial V}{\partial z}\hat{z}$$

$\vec{E}$ will be in z-dir^n only bcoz x & y axis are infinite.

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$$\vec{E} = -\frac{\partial}{\partial z}\left[\frac{q}{4\pi\epsilon_0\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{4\pi\epsilon_0\sqrt{x^2+y^2+(z+d)^2}}\right]\hat{z}$$

$$= -\frac{q}{4\pi\epsilon_0}\left[-\frac{1}{2}\left(x^2+y^2+(z-d)^2\right)^{-3/2}(2z-2d) + \frac{1}{2}\left(x^2+y^2+(z+d)^2\right)^{-3/2}(2z+2d)\right]$$

$$\boxed{\vec{E} = +\frac{q}{4\pi\epsilon_0}\left[\frac{(z-d)}{\left(x^2+y^2+(z-d)^2\right)^{3/2}} - \frac{(z+d)}{\left(x^2+y^2+(z+d)^2\right)^{3/2}}\right]\hat{z}}$$

This is the electric field at a z-distance above the plane.

"Electric field below the plane" $\boxed{\vec{E} = 0}$

Que :- $\sigma_{induce} = ?$

Total charge induce $q_{in} = \int \sigma_{in}\, da$

force, energy W

(JEST) What is the $\vec{E}$ at $(0,0,-d)$

(i) ∞ (ii) $-\infty$ (iii) 0 (iv) $\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$

bcoz at $\boxed{z<0,\ V=0\ \&\ E=0}$

but $\boxed{z>0,\ V\neq 0,\ E\neq 0}$

Induced surface charge density on the conducting surface :-

$$\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\epsilon_0}$$

$$V_{above} = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

$$V_{below} = 0$$

$$\sigma_{in} = -\epsilon_0 \left.\frac{\partial V_{above}}{\partial z}\right|_{z=0}$$

$$= -\epsilon_0\left[\frac{-d}{(x^2+y^2+d^2)^{3/2}} - \frac{d}{(x^2+y^2+d^2)^{3/2}}\right]\frac{q}{4\pi\epsilon_0}$$

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$$\boxed{\sigma_{in} = \frac{qd}{2\pi (x^2+y^2+d^2)^{3/2}}}$$

σ_{in} is non-uniform & negative.

Total charge induced, on the surface.

$$q' = \int_s \sigma \, da$$

we have to convert cartesian co-ordinate into spherical polar-coordinate system.

$$\vec{r} = x\hat{i} + y\hat{j} \Rightarrow r^2 = x^2+y^2$$

$$da = ds_\theta = r\,dr\,d\phi \quad (\text{bcoz } \theta \text{ is constant here})$$

$$q' = \int_0^\infty \int_0^{2\pi} \sigma \, r\,dr\,d\phi = \int_0^\infty \sigma\, r\,dr\,(2\pi)$$

$$= \int_0^\infty \frac{-qd}{\cancel{2\pi}(r^2+d^2)^{3/2}}\, r\,dr\,(\cancel{2\pi})$$

$$= -qd \int_0^\infty \frac{r\,dr}{(r^2+d^2)^{3/2}}$$

$r^2+d^2 = t$

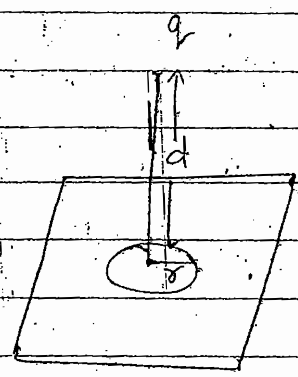
$2r\,dr = dt$

$$\int \frac{r\,dr}{(r^2+d^2)^{3/2}} = \int \frac{dt}{2t^{3/2}} = \frac{1}{2}\frac{t^{-1/2}}{-1/2} = -\frac{1}{t^{1/2}}$$

$$\text{So } q' = -qd\left[\frac{-1}{(r^2+d^2)^{1/2}}\right]_0^\infty = -qd\left[\frac{1}{\infty}+\frac{1}{d}\right]$$

$$\boxed{q' = -q}$$

Q. Total charge at infinite disc at distance r.



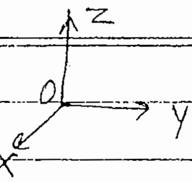
$$q' = \int_0^r -qd \frac{r\,dr}{(r^2+d^2)^{3/2}}$$

$$= -qd\left[\frac{1}{(r^2+d^2)^{1/2}}\right]_0^r$$

$$q' = -qd\left[-\frac{1}{(r^2+d^2)^{1/2}} + \frac{1}{d}\right]$$

Force :- force on charge $+q$,

The dir^n of force is towards $-q$ charge.



Matematically the image of $+q$ is $-q$. This image $-q$ will attract the charge but practically the surface attract it

$$\boxed{\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{4d^2}(-\hat{z})}$$

Energy :- Work done to bring charge q from ∞ to the point O.

Force is not variable along x & y dir^n.

It is variable along z-dir^n only

Variable force $F = \frac{-1}{4\pi\epsilon_0}\frac{q^2}{4z^2}$

$$W = \int_\infty^d F \cdot dr$$

$$W = \int_\infty^d \frac{1}{4\pi\epsilon_0}\frac{q^2}{4z^2}\,dz$$

$$W = \frac{1}{4\pi\epsilon_0}\frac{q^2}{4}\left[\frac{z^{-2+1}}{-2+1}\right]_\infty^d = \frac{1}{4\pi\epsilon_0}\frac{q^2}{4}\left[\frac{-1}{z}\right]_\infty^d$$

$(\frac{1}{d} - \frac{1}{\infty})$

$$\boxed{U = W = \frac{-1}{4\pi\epsilon_0}\frac{q^2}{4d}}$$

Work done will store in form of energy. This is the electrostatic energy.

Prob. 2 :- A point charge q is situated at a distance a from the centre of grounded conducting sphere of radius R. Find out the pot^n, electric field, induced charge, force of this configuration.

Inside & outside the surface Pot^n = 0 bcoz surfaces of sphere are equipot^n.

Pot^n remains constant but induced charge may or may not be 0.

Induced charge = Image charge

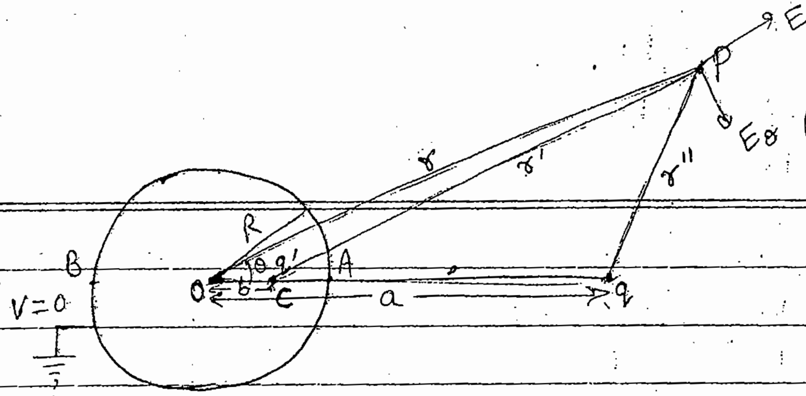


Image charge :- let induced charge q' is at pt C at a distance b from origin.

$$V_A = V_B = 0 \rightarrow \text{boundry cond}^n$$

$$V_A = \frac{q}{4\pi\epsilon_0(a-R)} + \frac{q'}{4\pi\epsilon_0(R-b)} = 0$$

$$\frac{q}{a-R} + \frac{q'}{R-b} = 0 \Rightarrow \frac{q}{a-R} = \frac{-q'}{R-b}$$

$$q(R-b) = -q'(a-R)$$

$$\boxed{q' = \frac{-q(R-b)}{(a-R)}}$$

$$V_B = \frac{q}{4\pi\epsilon_0(a+R)} + \frac{q'}{4\pi\epsilon_0(R+b)} = 0$$

$$\Rightarrow \frac{q}{a+R} = \frac{-q'}{R+b} \Rightarrow \boxed{q' = \frac{-q(R+b)}{(R+a)}}$$

Equating $q' = \frac{-q(R-b)}{(a-R)} = \frac{-q(R+b)}{(R+a)}$

$$\Rightarrow (R+a)(R-b) = (R+b)(a-R)$$

$$R^2 - \cancel{Rb} + \cancel{aR} - ab = \cancel{aR} - R^2 + ab - \cancel{bR}$$

$$R^2 - ab = -(R^2 - ab)$$

$$2R^2 = 2ab \Rightarrow R = \sqrt{ab}$$

$$\Rightarrow \boxed{b = \frac{R^2}{a}}$$

So $q' = \frac{-q\left(R - \frac{R^2}{a}\right)}{(a-R)} = \frac{-q(aR-R^2)}{a(a-R)}$

$$q' = -\frac{q(a-R)R}{(a-R)a} \Rightarrow \boxed{q' = \frac{-qR}{a}}$$

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Now, ~~Not~~ we can replace this whole conducting sphere by its image q'.

$r' \to r \,\&\, \theta \quad \& \quad r'' \to r \,\&\, \theta$

Pot^n at P, $V_P = \dfrac{q}{4\pi\epsilon_0 r''} + \dfrac{q'}{4\pi\epsilon_0 r'}$

from law of cosine, $(r'')^2 = r^2 + a^2 - 2ra\cos\theta$

$(r')^2 = r^2 + b^2 - 2rb\cos\theta$

{ −ve when Angle < 90°, +ve if > 90°, 0. if = 90°

$$V_P = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{(r^2+a^2-2ra\cos\theta)^{1/2}} - \frac{qR}{a}\frac{1}{(r^2+b^2-2br\cos\theta)^{1/2}}\right]$$

$$V_P = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{(r^2+a^2-2ra\cos\theta)^{1/2}} - \frac{R}{a\left(r^2+\frac{R^4}{a^2}-2r\frac{R^2}{a}\cos\theta\right)^{1/2}}\right]$$

At pt. A, $r = R$, $\theta = 0$

$$V_A = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{(R^2+a^2-2Ra)^{1/2}} - \frac{R}{a\left(R^2+\frac{R^4}{a^2}-2\frac{R^3}{a}\right)^{1/2}}\right]$$

$\left(R - \frac{R}{a}\right)^2$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{(R-a)^{2\times\frac{1}{2}}} - \frac{\cancel{R}\,1}{a\left(\cancel{R}1-\frac{R}{a}\right)^{2\times\frac{1}{2}}}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R-a} - \frac{1}{(R-a)}\right]$$

$\boxed{V_A = 0}$

& $\boxed{V_{in} = 0}$

Electric field:- $\vec{E_P} = -\vec{\nabla} V_P$

$\vec{E}$ depends upon r & θ, we get two dir^n of $\vec{E}$

$\vec{E_r}$ & $\vec{E_\theta}$

$$\vec{E_P} = -\frac{\partial V_P}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V_P}{\partial \theta}\hat{\theta}$$

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$$\frac{\partial V_P}{\partial r} = \frac{q}{4\pi\epsilon_0}\left[\frac{(r-a\cos\theta)}{(r^2+a^2-2ra\cos\theta)^{3/2}} - \frac{R(r-\frac{R^2}{a}\cos\theta)}{a(r^2+\frac{R^4}{a^2}-2r\frac{R^2}{a}\cos\theta)^{3/2}}\right]$$

$$\frac{1}{r}\frac{\partial V_P}{\partial \theta} = \frac{1}{r}\frac{q}{4\pi\epsilon_0}\left[\frac{-2ra\sin\theta}{(r^2+a^2-2ra\cos\theta)^{3/2}} - \frac{R\,\cancel{2}r\frac{R^2}{a}\sin\theta}{a(r^2+\frac{R^4}{a^2}-2r\frac{R^2}{a}\cos\theta)^{3/2}}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{-2a\sin\theta}{(r^2+a^2-2ra\cos\theta)^{3/2}} - \frac{2R^3\sin\theta}{a^2(r^2+\frac{R^4}{a^2}-2r\frac{R^2}{a}\cos\theta)^{3/2}}\right]$$

σ induced :- $\sigma_{in} = -\epsilon_0\left[\frac{\partial V_{above}}{\partial r} - \underbrace{\frac{\partial V_{below}}{\partial r}}_{0}\right]_{r=R}$

$$\sigma_{in} = -\epsilon_0 \left.\frac{\partial V_{above}}{\partial r}\right|_{r=R}$$

$$\sigma_{in} = -\epsilon_0\frac{q}{4\pi\epsilon_0}\left[\frac{(R-a\cos\theta)}{(R^2+a^2-2Ra\cos\theta)^{3/2}} - \frac{\frac{R}{a}(R-\frac{R^2}{a}\cos\theta)}{\frac{R^3}{a^3}(a^2+R^2-2Ra\cos\theta)^{3/2}}\right]$$

$$= \frac{-q}{4\pi(R^2+a^2-2Ra\cos\theta)^{3/2}}\left[R-a\cos\theta - \frac{a^2}{R^2}(R-\frac{R^2}{a}\cos\theta)\right]$$

$$= \frac{-q}{4\pi(R^2+a^2-2Ra\cos\theta)^{3/2}}\left[R-\cancel{a\cos\theta}-\frac{a^2}{R}+\cancel{a\cos\theta}\right]$$

$$= \frac{-q(R-a^2/R)}{4\pi r^3} \Rightarrow \sigma_{in} = \frac{-q(R^2-a^2)}{4\pi a r^3}$$

σ_{in} depends on r & θ so Non-uniform & -ve

Total Induced charge :- $q'' = \int_S \sigma\, ds_r$

$$q'' = \int_S \frac{-q(R^2-a^2)}{4\pi a(R^2+a^2-2Ra\cos\theta)^{3/2}} R^2\sin\theta\, d\theta\, d\phi$$

force :- Force on q is attracted bcoz in sphere induced charge is $-q$.

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a-b)^2}$$

$$F = \frac{-1}{4\pi\epsilon_0}\frac{q^2R}{a\left(a-\frac{R^2}{a}\right)^2}$$

$$\boxed{F = \frac{-q^2Ra}{4\pi\epsilon_0(a^2-R^2)^2}}$$ force will be attractive

or $$\boxed{\vec{F} = \frac{q^2Ra}{4\pi\epsilon_0(a^2-R^2)^2}(-\hat{r})}$$

Energy :- $W = -\int_\infty^a F\cdot d\vec{r}$

when we take q from ∞ then a will be variable then $a = r$

$$W = +\int_\infty^a \frac{q^2Rr}{4\pi\epsilon_0(r^2-R^2)^2}\,dr$$

Let $r^2 - R^2 = u$

$r\,dr = \frac{du}{2}$

$\int \frac{du}{2u^2} = \frac{-1}{2u}$

$$= \frac{q^2R}{4\pi\epsilon_0}\left[\frac{-1}{2(r^2-R^2)}\right]_\infty^a$$

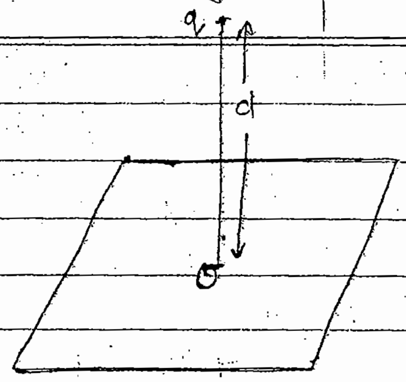
$$= \frac{-q^2R}{4\pi\epsilon_0(2)}\left[\frac{1}{a^2-R^2} - \frac{1}{\infty}\right]$$

$$\boxed{U = W = \frac{-q^2R}{8\pi\epsilon_0}\frac{1}{(a^2-R^2)}}$$

Prob :- A point charge q of mass m is released from the rest at a distance d from an ∞ grounded conducting plane. How long it will take to hit the plane.

Force is variable along z-dir^n

$$\vec{F} = \frac{-1}{4\pi\epsilon_0}\frac{q^2}{4z^2}$$



$$F = \frac{-1}{4\pi\epsilon_0}\frac{q^2}{4z^2} = m\frac{d^2z}{dt^2} = m\frac{dv}{dt}$$

Multiply by v on both sides

$$mv\left(\frac{dv}{dt}\right) = \frac{-1}{4\pi\epsilon_0}\frac{q^2}{4z^2}\cdot v$$

$$= \frac{-1}{4\pi\epsilon_0}\frac{q^2}{4z^2}\frac{dz}{dt}$$

$$m\frac{d}{dt}\left[\frac{1}{2}v^2\right] = \frac{q^2}{16\pi\epsilon_0}\frac{d}{dt}\left[\frac{1}{z}\right]$$

$$d\left[\frac{1}{2}v^2\right] = \frac{q^2}{16\pi\epsilon_0 m}\, d\left[\frac{1}{z}\right]$$

Integrating, $\frac{1}{2}v^2 = \frac{q^2}{16\pi\epsilon_0 m}\cdot\frac{1}{z} + C'$

At $z = d$, $v = 0$

$$\Rightarrow 0 = \frac{q^2}{16\pi\epsilon_0 m}\cdot\frac{1}{d} + C \Rightarrow C = -\frac{q^2}{16\pi\epsilon_0 m d}$$

So $\frac{1}{2}v^2 = \frac{q^2}{16\pi\epsilon_0 m}\left[\frac{1}{z} - \frac{1}{d}\right]$

$$\boxed{v^2 = \frac{q^2}{8\pi\epsilon_0 m}\left[\frac{1}{z} - \frac{1}{d}\right]}$$

by this velocity charge will hit the plane

$$\frac{dz}{dt} = \boxed{v = \sqrt{\frac{q^2}{8\pi\epsilon_0 m}\left[\frac{1}{z} - \frac{1}{d}\right]}}$$

$$\int_d^0 \frac{dz}{\left[\frac{1}{z} - \frac{1}{d}\right]^{1/2}} = \int_0^t \frac{q}{\sqrt{8\pi\epsilon_0 m}}\, dt$$

$$\text{L.H.S.} = \int_d^0 \frac{d^{1/2} z^{1/2}}{\sqrt{d-z}}\, dz \Rightarrow \text{Let } z = u^2 \Rightarrow dz = 2u\,du$$

$$= \int_{\sqrt{d}}^0 \frac{d^{1/2} u \; 2u\,du}{\sqrt{d-u^2}} = 2d^{1/2}\int_{\sqrt{d}}^0 \frac{u^2\,du}{\sqrt{(\sqrt{d})^2-u^2}}$$

$$= 2d^{1/2}\left[-\frac{u}{2}\sqrt{d-u^2} + \frac{\sqrt{d}}{2}\sin^{-1}\frac{u}{\sqrt{d}}\right]_{\sqrt{d}}^0$$

{standard integral}

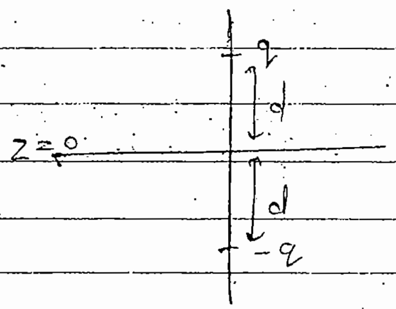
$$= 2 d^{1/2}\left[-\frac{d}{2}\,\frac{\pi}{2}\right] = -d\,\frac{\pi}{2}$$

So $\quad -\dfrac{d\pi}{2} = \dfrac{q}{\sqrt{8\pi\epsilon_0 m}}\, t \quad$ (time can never be -ve)

$$t = \frac{\pi d}{2q}\sqrt{8\pi\epsilon_0 m}$$

$$\boxed{t = \frac{\pi d\sqrt{2\pi\epsilon_0 m}}{q}}$$

Note :- In Prob-1 We can find out energy by this formula



$$W = \sum_{i=1}^{2}\sum_{j=1}^{2}\frac{1}{8\pi\epsilon_0}\frac{q_i q_j'}{r_{ij}}$$

$$= \frac{1}{8\pi\epsilon_0}\left(-\frac{q^2}{2d} - \frac{q^2}{2d}\right)$$

$$= \frac{1}{8\pi\epsilon_0}\left(-\frac{q^2}{d}\right)$$

Now Multiply by $\frac{1}{2}$ bcoz there is one real charge & 1 is its image & Energy will be due to real charge Not by its image.

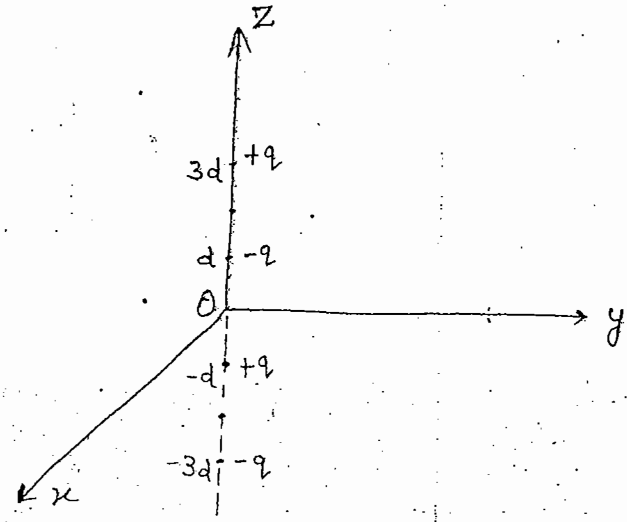
$$W = \frac{1}{2}\left[\frac{1}{8\pi\epsilon_0}\left(-\frac{q^2}{d}\right)\right] = \frac{1}{16\pi\epsilon_0}\left(-\frac{q^2}{d}\right)$$

$$\boxed{W = -\frac{1}{4\pi\epsilon_0}\,\frac{q^2}{4d}}$$

28/7/2012

Q1 :- Find the force on a charge +q if x-y plane is a grounded conductor. Also find Energy of this configuratio

Firstely make the image charge setup.



$$\vec{F}_1 = \frac{+1}{4\pi\epsilon_0}\frac{q^2(-\hat{z})}{4d^2} + \frac{1}{4\pi\epsilon_0}\frac{q^2}{16d^2}(\hat{z}) + \frac{1}{4\pi\epsilon_0}\frac{q^2}{36d^2}(-\hat{z})$$

$$= \frac{-q^2}{4\pi\epsilon_0 d^2}\left[\frac{1}{4} - \frac{1}{16} + \frac{1}{36}\right]\hat{z}$$

$$= \frac{-q^2}{4\pi\epsilon_0 d^2}\left[\frac{36 - 9 + 4}{144}\right]\hat{z}$$

$$\boxed{\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}\left(-\frac{31}{144}\hat{z}\right)}$$

Energy of the configuration,

$$W_1 = -\int_{\infty}^{3d} F\cdot dr$$

$$= -\int_{\infty}^{3d}\frac{1}{4\pi\epsilon_0}\frac{q^2}{z^2}\left(\frac{-31}{144}\right)dz$$

[d = z bcoz when we take the charge from ∞ to that pt. then d is not fixed]

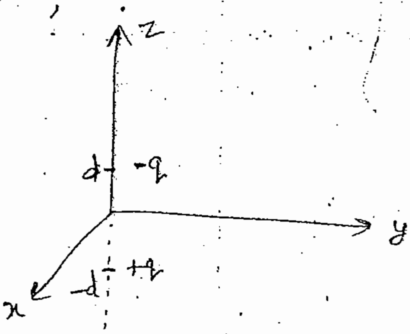
$$= \frac{1}{4\pi\epsilon_0}\frac{q^2\,31}{144}\left[\frac{-1}{z}\right]_{\infty}^{3d}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{31q^2}{144}\left(\frac{-1}{3d}\right)$$

$$\boxed{W_1 = \frac{-31\,q^2}{4\pi\epsilon_0(432d)}}$$

Work done to take charge -q from ∞,

force on -q charge,



$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{4d^2}(-\hat{z})$$

$$W_2 = -\int_{\infty}^{d} F\cdot dr$$

$$= -\int_{\infty}^{d}\frac{-1}{4\pi\epsilon_0}\frac{q^2}{4z^2}dz$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q^2}{4}\left[\frac{-1}{z}\right]_{\infty}^{d} \Rightarrow \boxed{W_2 = -\frac{q^2}{4\pi\epsilon_0(4d)}}$$

Total Energy of this system is

$$W = W_1 + W_2$$

$$\Rightarrow W = -\frac{31 q^2}{4\pi\epsilon_0 (432 d)} - \frac{q^2}{4\pi\epsilon_0 (4d)}$$

$$W = -\frac{q^2}{4\pi\epsilon_0 d}\left(\frac{31}{432} + \frac{1}{4}\right)$$

$$\boxed{W = \frac{-q^2}{4\pi\epsilon_0}\left(\frac{139}{432 d}\right)}$$

The Energy will be due to real charges only. We are not working on image charges.

We can find the Energy by the following formula also :-

$$\boxed{W' = \frac{1}{8\pi\epsilon_0} \sum_{\substack{i=1 \\ i \neq j}}^{4} \sum_{j=1}^{4} \frac{q_i q_j}{r_{ij}}}$$

but After getting the value of W just multiply by $\frac{1}{2}$ bcoz there are only 2 real charges.

$$\text{Total } W = \frac{1}{2} W'$$

Q.2 :- Two semi infinite grounded conducting plane meat at right angle. In a region b/w them, There is a point charge q. set up the image configuration & calculate

(1) Pot^n in the region

(2) Electric field

(3) force on the charge

(4) Energy of this configuration

No. of image charges depends upon the angle b/w the planes.

$$\boxed{n = \left(\frac{360^\circ}{\theta} - 1\right)}$$

$\theta \rightarrow$ angle b/w planes

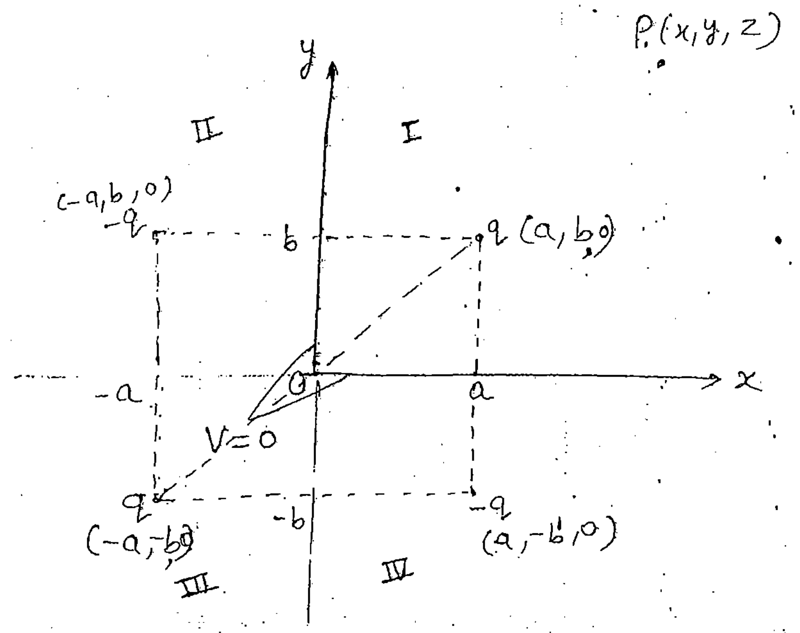
In present case $\theta = 90^\circ$

so $n = \left(\frac{360}{90} - 1\right) = (4-1)$

$n = 3$ So there will be 3 image charges.

Potential in Quardrate I,

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{(x-a)^2+(y-b)^2+z^2}} - \frac{q}{\sqrt{(x+a)^2+(y-b)^2+z^2}} + \frac{q}{\sqrt{(x+a)^2+(y+b)^2+z^2}} - \frac{q}{\sqrt{(x-a)^2+(y+b)^2+z^2}}\right]$$



We know that, at $z=0$

Potⁿ $V=0$ {cheak the potⁿ by this B.C.}

Potⁿ in Quardrate II, III & IV will be

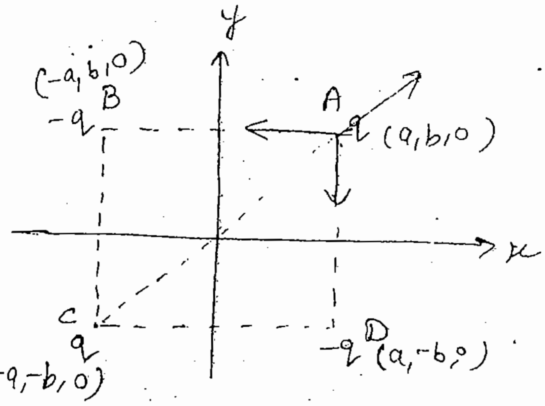
$$\boxed{V_{II} = 0 \quad , \; V_{III} = 0 \quad , \; V_{IV} = 0}$$

<u>Electric field</u> :- $\vec{E_I} = -\vec{\nabla} V$

$$\boxed{E_{II} = E_{III} = E_{IV} = 0}$$

$$\vec{E_I} = -\left[\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right]$$

Force :- $\vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^3}\vec{r}$



force on charge q is

$$\vec{F}_{CA} = \frac{q^2}{4\pi\epsilon_0 |CA|}\vec{CA}$$

$$\vec{F}_{CA} = \frac{q^2 (a\hat{x} + b\hat{y})}{4\pi\epsilon_0\, 4(a^2+b^2)^{3/2}} \quad \text{——(1)}$$

$\{ \vec{CA} = \vec{r} = 2a\hat{x} + 2b\hat{y}$ & $|\vec{r}| = r = 2\sqrt{a^2+b^2} \}$

$$\vec{F}_{AD} = \frac{+q^2}{4\pi\epsilon_0 (4b^2)}(-\hat{y}) \quad \text{——(2)}$$

$$\vec{F}_{AB} = \frac{q^2}{4\pi\epsilon_0 (4a^2)}(-\hat{x}) \quad \text{——(3)}$$

Total force on q, $\vec{F} = \vec{F}_{CA} + \vec{F}_{AD} + \vec{F}_{AB}$

$$\boxed{\vec{F} = \frac{q^2}{16\pi\epsilon_0}\left[\underbrace{\left(\frac{a}{(a^2+b^2)^{3/2}} - \frac{1}{a^2}\right)\hat{x}}_{\text{force in x-dir}^n} + \underbrace{\left(\frac{b}{(a^2+b^2)^{3/2}} - \frac{1}{b^2}\right)\hat{y}}_{\text{force in y-dir}^n}\right]}$$

If $\boxed{a=b}$ then

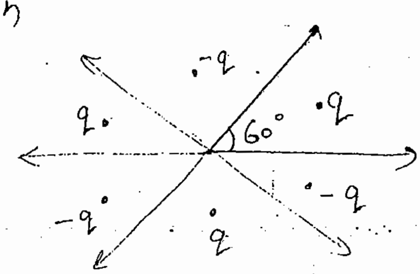
$$\vec{F} = \frac{q^2}{16\pi\epsilon_0}\left[\left(\frac{1}{2\sqrt{2}a^2} - \frac{1}{a^2}\right)\hat{x} + \left(\frac{1}{2\sqrt{2}a^2} - \frac{1}{a^2}\right)\hat{y}\right]$$

Note :- If $\angle$ b/w plane is $60°$ then

$n = \left(\frac{360}{60} - 1\right) = (6-1)$

$n = 5$

There will be 5 images alternate $+q$ & $-q$.



This charge set-up is only for odd No. of Image charges.

$\boxed{\text{No. of images} = \text{Odd}}$

Q.3 :- The Electric potⁿ of a grounded conducting sphere of radius R in a uniform electric field $E_0 \hat{z}$ is given by

$$\phi(r,\theta) = -E_0 r\cos\theta + \frac{E_0 R^3 \cos\theta}{r^2} \quad \text{—(1)}$$

where, r is the distance from the centre & θ is the angle b/w r & z dirⁿ.

(a) What is the dipole moment acquired by the sphere.

(b) Calculate radial & tangential comp. of electric field on the surface of sphere.

(c) What is the surface charge density induced on the sphere.

(d) Total charge induced on the sphere.

option (a) is on HOLD.

In $\phi \rightarrow$ II term is dipole term $\left(V_{dipole} \propto \frac{1}{r^2}\right)$

$$V_{dipole} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2} \quad \text{—(2)}$$

compare (1) & (2) $\Rightarrow$ $\boxed{\vec{p} = 4\pi\epsilon_0 E_0 R^3 \hat{z}}$

(b) $\vec{E} = -\vec{\nabla} V$

{ at surface $r = R$

Radial comp. $\vec{E_r} = \dfrac{-\partial V}{\partial r} = -\dfrac{\partial \phi}{\partial r}\Big|_{r=R}$

$$\vec{E_r} = \frac{-\partial}{\partial r}\left[-E_0 r\cos\theta + \frac{E_0 R^3\cos\theta}{r^2}\right]_{r=R}$$

$$= -\left[-E_0\cos\theta + E_0 R^3\cos\theta(-2r^{-3})\right]_{r=R}$$

$$= \left|E_0\cos\theta + \frac{2E_0 R^3\cos\theta}{r^3}\right|_{r=R}$$

$$= E_0\cos\theta\left(1 + \frac{2R^3}{r^3}\right) = E_0\cos\theta\,[1+2] \Rightarrow \boxed{\vec{E_r} = 3E_0\cos\theta}$$

Tangential comp.

$$\vec{E_\theta} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{1}{r}\frac{\partial}{\partial\theta}\left[-E_0 r\cos\theta + \frac{E_0 R^3\cos\theta}{r^2}\right]_{r=R}$$

$$= \frac{-1}{r}\left[+E_0 r\sin\theta - \frac{E_0 R^3}{r^2}\sin\theta\right]_{r=R}$$

$$= \left[-E_0\sin\theta + E_0\frac{R^3}{r^3}\sin\theta\right]_{r=R} = -E_0\sin\theta + E_0\sin\theta$$

$$\boxed{\vec{E_\theta} = 0}$$

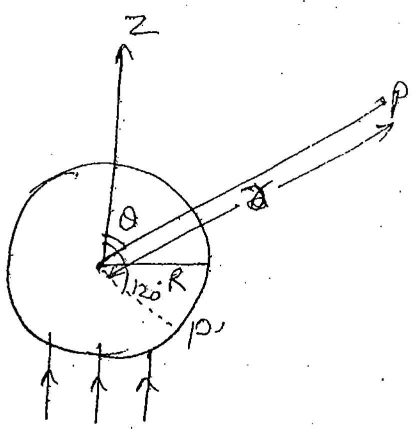
Electric field on the surface is entirely normal to the surface.

On surface, tangential comp is always 0.

(C) Surface charge density induced,

boundary condⁿ hold on surface

$$\left[\frac{\partial V_{above}}{\partial r} - \frac{\partial V_{below}}{\partial r}\right]_{r=R} = -\frac{\sigma_{in}}{\epsilon_0}$$



$\because$ $V_{below} = 0$

$$\therefore \sigma_{in} = -\epsilon_0 \left.\frac{\partial \phi_{above}}{\partial r}\right|_{r=R}$$

$$\sigma_{in} = -\epsilon_0(-3E_0\cos\theta)$$

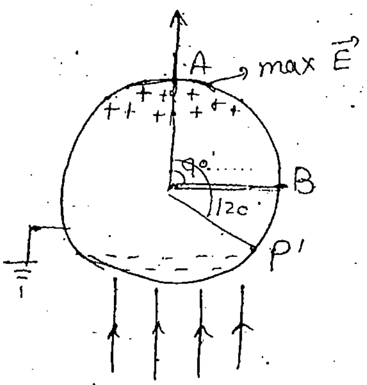
$$\boxed{\sigma_{in} = +3\epsilon_0 E_0 \cos\theta}$$

It is varying with θ so σ_{in} is Non-uniform.

* surface charge density at P' is

$$\sigma_{in} = 3\epsilon_0 E_0 \cos 120°$$

$$\sigma_{in} = 3\epsilon_0 E_0\left(\frac{-1}{2}\right) = \frac{-3}{2}\epsilon_0 E_0$$



At A, $\cancel{3\epsilon}$ $\sigma_{in} = 3\epsilon_0 E_0 \cos(0°)$

$$\boxed{\sigma_{in} = 3\epsilon_0 E_0} \text{ (maxi)}$$

At B, $\sigma_{in} = 3\epsilon_0 E_0 \cos 90°$

$$\boxed{\sigma_{in} = 0} \text{ (min.)}$$

{∠ → is b/w z-dirⁿ & $\vec{E}$ }

(d) Total charge induced

$$q = \int \sigma \, da$$

$$q = \int_0^{\pi}\int_0^{2\pi} 3\epsilon_0 E_0 \cos\theta \, R^2 \sin\theta \, d\theta \, d\phi$$

$$q = 3\epsilon_0 E_0 R^2 [2\pi] \underbrace{\int_0^{\pi} \sin\theta\cos\theta \, d\theta}_{=0}$$

$$q = 0$$

charge on whole sphere = 0

upper hemisphere = +ve

lower = −ve

charge on upper hemisphere is

$$q = \int \sigma \, da = \int_0^{\pi/2}\int_0^{2\pi} R^2 3\epsilon_0 E_0 \sin\theta\cos\theta \, d\theta \, d\phi$$

$$= R^2 3\epsilon_0 E_0 \int_0^{\pi/2} \frac{\sin 2\theta}{2}\, d\theta\, [2\pi]$$

$$= \frac{2\pi R^2 3\epsilon_0 E_0}{2} \left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$q = 3\pi R^2 \epsilon_0 E_0 \left[\frac{-(-1)+1}{2}\right] = \boxed{3\pi R^2 \epsilon_0 E_0}$$

charge on lower hemisphere is,

$$\boxed{q = -3\pi R^2 \epsilon_0 E_0} \qquad [\theta : \pi/2 \to 0]$$

Laplace Equation :-

$$\boxed{\nabla^2 V = 0} \quad \text{In 3 Dim.}$$

In Gauss, Coulomb's law $\Rightarrow$ ρ, q (given)

$\downarrow$

V, E (find)

But Here In laplace eqn $\Rightarrow$ V (given)

$\downarrow$

$E \to \rho, q$

We can use laplace eqn in 1 D, 2D & 3D. But In 2D & 3D problems becoms complicated.

So We solve Laplace eqn only in 1 Dim.

Q.1 :- Consider the parallel conductor (parallel plate capacitor) as shown in the figure where $V = 0$ at $z = 0$ & $V = 100\,V$ at $z = d$. Assuming the region b/w the plates charge free. Calculate Potential & Electric field b/w the plates. Also find charge densities on the plates. (No charge $\to$ Gauss law ✗)

$\rho = 0$ bcoz the region b/w the plates is charge free.

Laplace eqn is $\nabla^2 V = 0$

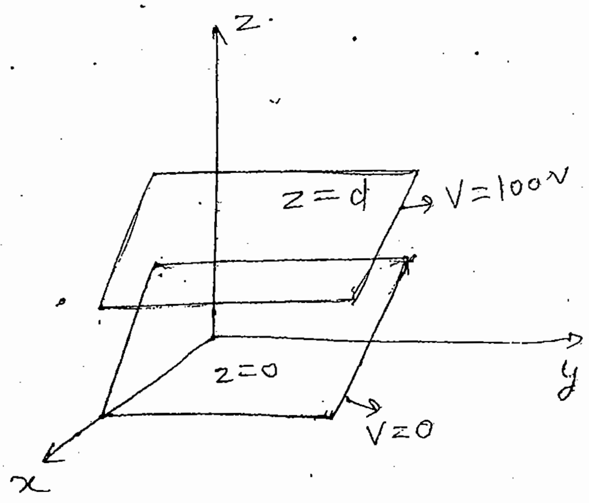
In 1 Dim $\frac{\partial^2 V}{\partial z^2} = 0$

Variation of potn is in z-dirn only.

$$\frac{\partial^2 V}{\partial z^2} = 0$$

$$\Rightarrow \frac{\partial V}{\partial z} = a$$

$$\boxed{V = az + b} \quad \text{—— (1)}$$



At $z = 0$, $V = 0$

So $\boxed{0 = b}$

@ $z = d$, $V = 100\,V$

$\Rightarrow 100 = ad + 0$

$$\boxed{a = \frac{100}{d}}$$

So $$\boxed{V = \frac{100\,z}{d} \text{ volt}} \quad \text{—— (2)}$$

This potⁿ is correct for b/w the plates. Not correct for outside the plates.

Electric field :-

$$\vec{E} = -\vec{\nabla} V$$

$$\vec{E} = -\frac{\partial}{\partial z}\left[\frac{100\,z}{d}\right]$$

$$\boxed{\vec{E} = -\frac{100}{d}\hat{z}}$$

Dirⁿ of $\vec{E}$ → downwards [from heigher to lower potⁿ]

Charge density :-

$\sigma_{upper} = ?$

for upper plate

$V_{above} = 0$

$V_{below} = \frac{100z}{d}$

$$\left.\frac{\partial V_{above}}{\partial z}\right|_{z=d} - \left.\frac{\partial V_{below}}{\partial z}\right|_{z=d} = -\frac{\sigma_{upper}}{\epsilon_0}$$

$$0 - \left.\frac{100}{d}\right|_{z=d} = -\frac{\sigma_{upper}}{\epsilon_0}$$

$$\boxed{\sigma_{upper} = \epsilon_0 \frac{100}{d} \ C/m^2}$$

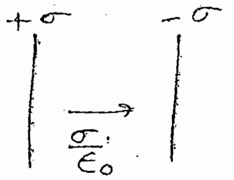
$\sigma_{lower} = ?$ @ $V_{above} = \frac{100z}{d}$ & $V_{below} = 0$

$$\left.\frac{100}{d}\right|_{z=d} = -\frac{\sigma_{lower}}{\epsilon_0}$$

$$\boxed{\sigma_{lower} = -\frac{100\,\epsilon_0}{d} \; \frac{C}{m^2}}$$

Verify

By Gauss law, $\oint E\cdot ds = \frac{q}{\epsilon_0} = \frac{\int \sigma\, ds}{\epsilon_0} \Rightarrow \vec{E} = \frac{\sigma}{\epsilon_0}$



$$\vec{E} = \frac{100}{d}(-\hat{z})$$

Qus.2:- Consider 2 concentric spherical conducting shell at the origin. The outer radius of the inner shell is r_a & inner radius of the outer shell is r_b. If charge density $\rho = 0$ for $r_a < r < r_b$ {spherical capacitor}

$V = 0$ at $r = r_a$

$V = V_0$ at $r = r_b$

Find (i) Potⁿ in the region $r_a < r < r_b$

(ii) Electric field " "

(iii) σ_a & σ_b

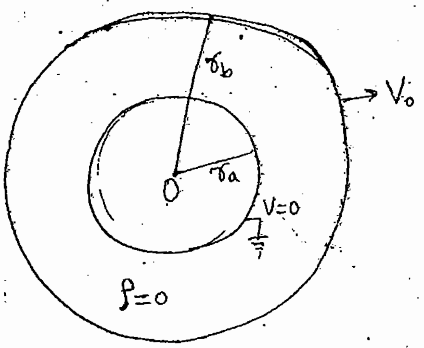
(iv) Q_a & Q_b

(V) Capacitance

Here potⁿ is varying with r. Not with θ & ϕ. So

Laplace eqⁿ

$$\nabla^2 V = 0$$



$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

$$\Rightarrow r^2 \frac{\partial V}{\partial r} = A \Rightarrow \frac{\partial V}{\partial r} = \frac{A}{r^2}$$

Integrate both sides, $\boxed{V = -\frac{A}{r} + B}$ —(1)

At $r = r_a$, $V = 0 \Rightarrow 0 = -\frac{A}{r_a} + B \Rightarrow \boxed{B = \frac{A}{r_a}}$

At $r = r_b$, $V = V_0 \Rightarrow V_0 = -\frac{A}{r_b} + \frac{A}{r_a} = A\left(-\frac{1}{r_b} + \frac{1}{r_a}\right)$ —(2)

$= A\left(\frac{-r_a + r_b}{r_a r_b}\right)$

$$\Rightarrow \boxed{A = \frac{V_0 (r_a r_b)}{(r_b - r_a)}}$$ —(3)

$$\therefore\ B = \frac{A}{r_a} = \frac{1}{r_a}\left[\frac{V_0 r_a r_b}{(r_b - r_a)}\right] \Rightarrow \boxed{B = \frac{r_b V_0}{(r_b - r_a)}} \quad \text{---}(4)$$

So $$V = -\frac{V_0 r_a r_b}{r(r_b - r_a)} + \frac{V_0 r_b}{(r_b - r_a)} \quad \text{---}(5)$$

This is the pot^n b/w 2 shells. It satisfies the B.C.

<u>Electric field</u> :- $\vec{E} = -\vec{\nabla} V = -\frac{\partial V}{\partial r}\hat{r}$

$$\vec{E} = \frac{V_0 r_a r_b}{r_b - r_a}\left(\frac{-1}{r^2}\right)\hat{r}$$

$$\vec{E} = -\frac{V_0 r_a r_b}{r^2(r_b - r_a)}\hat{r}$$

Dir^n of $\vec{E}$ is $-\hat{r}$ bcoz of higher to lower pot^n.

Now, <u>σ_a & σ_b</u> :- for σ_a, $\qquad V_{below} = 0$

$$\frac{\partial V_{above}}{\partial r} - \frac{\partial V_{below}}{\partial r} = -\frac{\sigma_a}{\epsilon_0}$$

$$\sigma_a = -\epsilon_0 \left.\frac{\partial V_{above}}{\partial r}\right|_{r=r_a}$$

$$\sigma_a = -\epsilon_0 \left.\frac{V_0 r_a r_b}{r^2(r_b - r_a)}\right|_{r=r_a} = -\epsilon_0 \frac{V_0 r_a r_b}{r_a^2(r_b - r_a)}$$

$$\boxed{\sigma_a = -\frac{\epsilon_0 V_0 r_b}{r_a(r_b - r_a)}}$$

for σ_b, $\quad V_{below} = -\frac{V_0 r_a r_b}{r(r_b - r_a)} + \frac{V_0 r_b}{(r_b - r_a)}$

$V_{above} = V_0 \qquad \left\{\frac{\partial V_0}{\partial r} = 0\right\}$

so $$\sigma_b = +\epsilon_0 \left.\frac{\partial V_{below}}{\partial r}\right|_{r=r_b}$$

$$\boxed{\sigma_b = \frac{\epsilon_0 V_0 r_a}{r_b(r_b - r_a)}}$$

σ_a & σ_b are uniform.

Q_a & Q_b :-

$$Q_a = \sigma_a \cdot 4\pi r_a^2$$

$$\boxed{Q_a = \frac{-4\pi \epsilon_0 V_0 r_a r_b}{(r_b - r_a)}}$$

$$Q_b = \sigma_b \cdot 4\pi r_b^2 \Rightarrow \boxed{Q_b = \frac{4\pi \epsilon_0 V_0 r_a r_b}{(r_b - r_a)}}$$

Q_a & Q_b are equal & opposite due to Induction.

Capacitance :-

Capaciter is a combination of two conductors held at different potentials. One is given charge $+q$ & other is given charge $-q$.

At higher potⁿ → charge will be $+q$
lower " " $-q$

bcoz $\vec{E}$ will be highr to lower potⁿ (or from $+q$ to $-q$)

Capacitance is defined as

$$\boxed{C = \frac{q}{V}}$$

where, q → charge for the +ve conductors.
V → potⁿ difference b/w two conductors

Capacitance is purely a geometrical quantity. It does not depend on charge & potential but depends on shape, size & geometrical placing of conductors & medium b/w them.

In previous Problem : Parallel Plate Capacitor

Potⁿ diffrence $\Rightarrow V = 100 - 0 \Rightarrow V = 100$ Volt

If plate area $= A$

charge on +ve conductor $\quad q_+ = \sigma A$

$$q_+ = \frac{100\,\epsilon_0 A}{d}\ C$$

$$\boxed{C = \frac{q_+}{100} = \frac{\epsilon_0 A}{d}}\ \text{Farad}$$

Now for spherical plate Capacitor,

$$V = V_0 - 0 \Rightarrow V = V_0$$

$$Q_+ = Q_b = \frac{4\pi\epsilon_0 r_a r_b V_0}{(r_b - r_a)}$$

$$C = \frac{Q_+}{V_0} \Rightarrow \boxed{C = \frac{4\pi\epsilon_0 r_a r_b}{(r_b - r_a)}} \text{ Farad}$$

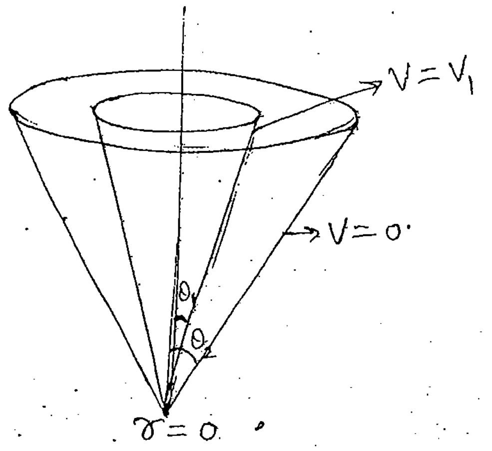
Capacitance of Isolated spherical plate capacitor :-

If r_b become ∞ then it become isolated spherical capacitor. bcoz there is no interaction of inner shell r_a & outer shell r_b.

At $r_b \to \infty$, $\boxed{C = 4\pi\epsilon_0 r_a}$

Earth is a Capacitor. find the Capacitance of Earth?
Earth is a isolated spherical capacitor. Capacitance is the capacity to hold charge. Current flow due to the pot^n difference.

Q.3:- ~~solom~~ solve Laplace eqn for the region b/w co-axial cones as shown in figure. Potn V_1 is assumed at θ_1 & $V=0$ at θ_2. The cone vertices are insulated at $r=0$.



Potn is constant with r & ϕ. Only variable with θ. Now Laplace Eqn, $\nabla^2 V = 0$

$$\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}\left(r \sin\theta \frac{\partial V}{\partial\theta}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0 \Rightarrow \sin\theta \frac{\partial V}{\partial\theta} = A$$

$$\frac{\partial V}{\partial\theta} = \frac{A}{\sin\theta} = A \operatorname{cosec}\theta$$

$$\boxed{V = A \ln\left(\tan\frac{\theta}{2}\right) + B} \quad \text{---(1)}$$

At $\theta = \theta_2$, $V = 0$ $\Rightarrow 0 = A \ln\left(\tan\frac{\theta_2}{2}\right) + B$

$$\Rightarrow \boxed{B = -A \ln\left(\tan\frac{\theta_2}{2}\right)} \quad \text{---(2)}$$

At $\theta = \theta_1$, $V = V_1 \Rightarrow$ $V_1 = A \ln\left(\tan\frac{\theta_1}{2}\right) - A \ln\left(\tan\frac{\theta_2}{2}\right)$

$$\Rightarrow V_1 = A \ln\left[\frac{\tan \theta_1/2}{\tan \theta_2/2}\right]$$

$$\Rightarrow \boxed{A = \frac{V_1}{\ln\left[\frac{\tan\frac{\theta_1}{2}}{\tan\frac{\theta_2}{2}}\right]}} \quad \text{---(3)}$$

Now $B = -A \ln\left(\tan\frac{\theta_2}{2}\right) = -\frac{V_1}{\ln\left[\frac{\tan\frac{\theta_1}{2}}{\tan\frac{\theta_2}{2}}\right]} \ln\left[\tan\frac{\theta_2}{2}\right]$ ---(4)

So $$\boxed{V = \frac{V_1 \ln(\tan\theta/2) - \ln(\tan\theta_2/2)}{\ln(\tan\frac{\theta_1}{2}) - \ln(\tan\frac{\theta_2}{2})}} \quad \text{——} (5)$$

This is the pot^n in b/w the cones.

Electric field :- $\quad \vec{E} = -\vec{\nabla} V$

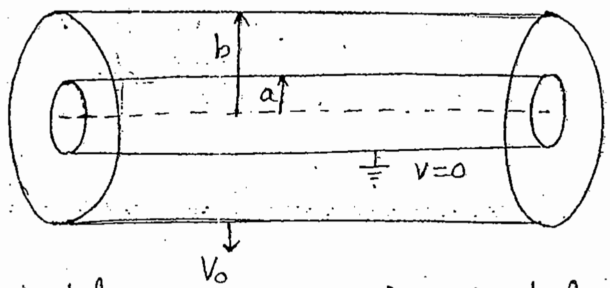
$$\vec{E} = -\frac{1}{r}\frac{\partial V}{\partial \theta}\,\hat{\theta}$$

$$\vec{E} = -\frac{1}{r}\left[\frac{\frac{V_1}{\tan\theta/2}\cdot \sec^2\frac{\theta}{2}\cdot\frac{1}{2} - \cancel{\ln(\tan\frac{\theta_2}{2})}}{\ln(\tan\frac{\theta_1}{2}) - \ln(\tan\frac{\theta_2}{2})}\right]\cdot\hat{\theta}$$

$$= \frac{-\frac{1}{r}\,\frac{V_1}{2}\,\frac{\cancel{\cos\theta/2}}{\sin\theta/2}\cdot\frac{1}{\cos^2\theta/2}}{\ln(\tan\frac{\theta_1}{2}) - \ln(\tan\frac{\theta_2}{2})}\,\hat{\theta}$$

$$\boxed{\vec{E} = \frac{-V_1}{r\sin\theta\left[\ln(\tan\frac{\theta_1}{2}) - \ln(\tan\frac{\theta_2}{2})\right]}\,\hat{\theta}}$$

Q.4:- Two coaxial concentric cylenders of radius a & b ($a<b$) if the inner one is grounded & outer is at pot^n V_0. Solve the Laplace eq^n to find (i) pot^n (ii) Electric field in b/w the cylenders. (iii) Capacitance of this combination.



Pot^n is variable with s so Laplace eq^n

$$\nabla^2 V = 0$$

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = 0 \Rightarrow \frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = 0$$

$$s\frac{\partial V}{\partial s} = A \Rightarrow \frac{\partial V}{\partial s} = \frac{A}{s}$$

$$\boxed{V = A\ln(s) + B} \quad \text{——(1)}$$

At $s = a$, $V = 0$ $\Rightarrow$ $0 = A\ln a + B$ $\Rightarrow$ $\boxed{B = -A\ln a}$

$s = b$, $V = V_0$ $\Rightarrow$ $V_0 = A\ln b - A\ln a = A\ln\frac{b}{a}$

$$\Rightarrow \boxed{A = V_0/\ln\frac{b}{a}} \quad \text{——(3)}$$

So $B = -A\ln a$

$$\boxed{B = \frac{-V_0\ln a}{\ln(b/a)}} \quad \text{——(4)}$$

So, $V = \frac{V_0}{\ln\frac{b}{a}}\ln s - \frac{\ln a}{\ln b/a}$

$$\boxed{V = \frac{V_0\ln(s/a)}{\ln(b/a)}} \quad \text{——(5)}$$

Electric field :- $\vec{E} = -\vec{\nabla}V$

$$= -\frac{\partial V}{\partial s}\hat{s}$$

$$= -V_0\frac{\frac{1}{s/a}\cdot\frac{1}{a}}{\ln b/a}\hat{s}$$

$$\boxed{\vec{E} = \frac{-V_0}{s\ln(b/a)}\hat{s}}$$

dir^n of $\vec{E} \rightarrow -\hat{s}$ (higher pot^n to lower pot^n)

Capacitance :- $C = \frac{Q}{V}$

$V =$ Pot^n diff $= V_0 - 0 \Rightarrow V = V_0$

To find Q, first find charge density of +ve conductor (i.e. ~~condu~~ cylinder of rad. b).

for Inner surface of outer cylinder

$$\left.\frac{\partial V_{above}}{\partial S}\right|_{S=b} \neq \left.\frac{\partial V_{below}}{\partial S}\right|_{S=b} = \frac{-\sigma_b}{\epsilon_0}$$

$V_{above} = V_0$ & $\frac{\partial V_0}{\partial s} = 0$

$$\Rightarrow \sigma_b = \frac{\epsilon_0 V_0}{b \ln(b/a)}$$

This is the charge density of outer cylender. It is Uniform.

charge $Q_b = \int \sigma_b \cdot da = \frac{\epsilon_0 V_0}{b \ln(b/a)} \cdot 2\pi bl$

$$\boxed{Q_b = \frac{\epsilon_0 V_0 2\pi l}{\ln(b/a)}}$$

Now, $C = \frac{Q_b}{V} \Rightarrow \boxed{C = \frac{2\pi\epsilon_0 l}{\ln(b/a)}}$

In general, Capacitance per unit length,

$$\boxed{C' = \frac{C}{l} = \frac{2\pi \epsilon_0}{\ln(b/a)}}$$

Que(1):- The plates of parallel plate capacitor are located at $x=0$ & $x=L$. The plate at $x=0$ is grounded while plate at $x=L$ is at potn V_0. The space b/w the plate has a uniform charge density ρ. Find the potn & Elec. field b/w the plates.

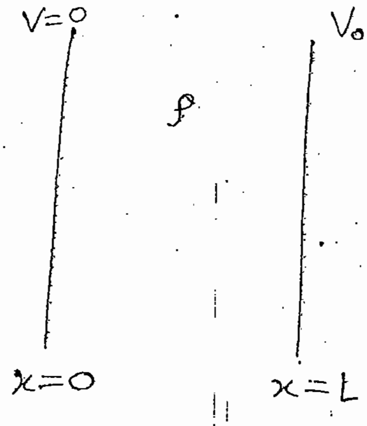
Que(2):- If, in the last problem $\boxed{\rho = -\frac{\rho_0 x}{L}}$ & $E=0$ at $x=0$ & $V=0$ at $x=L$. Find V & E b/w the plates.

Que(3):- Semi Infinite conducting planes at $\phi=0$ & $\phi=\pi/6$ are seperated by an small insulating gap as shown in the figure if $V(\phi=0)=0$ & $V(\phi=\pi/6)=100$ volt. Find V & E b/w the plates.

(1) By Poision's eq^n

$$\nabla^2 V = -\rho/\epsilon_0$$

$$\frac{d^2V}{dx^2} = -\frac{\rho}{\epsilon_0}$$



$$\Rightarrow \frac{dV}{dx} = -\frac{\rho x}{\epsilon_0} + A$$

$$\Rightarrow V = -\frac{\rho}{\epsilon_0}\frac{x^2}{2} + Ax + B \quad \text{—(1)}$$

At $x=0, V=0$, $\Rightarrow 0 = 0 + 0 + B \Rightarrow \boxed{B=0}$

At $x=L$, $V=V_0 \Rightarrow$

$$V_0 = -\frac{\rho}{\epsilon_0}\frac{L^2}{2} + AL + 0 \Rightarrow AL = V_0 + \frac{\rho_0 L^2}{2\epsilon_0}$$

$$\Rightarrow \boxed{A = \frac{V_0}{L} + \frac{\rho_0 L}{2\epsilon_0}}$$

So $$V = -\frac{\rho x^2}{2\epsilon_0} + \left(\frac{V_0}{L} + \frac{\rho_0 L}{2\epsilon_0}\right)x$$

or $$\boxed{V = -\frac{\rho x^2}{2\epsilon_0} + \frac{V_0 x}{L} + \frac{\rho_0 L x}{2\epsilon_0}}$$

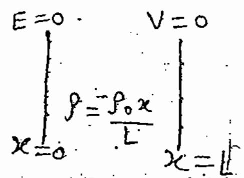
Now $\vec{E} = -\vec{\nabla} V = -\frac{\partial V}{\partial x}\hat{x}$

$$= -\frac{\partial}{\partial x}\left[-\frac{\rho x^2}{2\epsilon_0} + \frac{V_0 x}{L} + \frac{\rho_0 L x}{2\epsilon_0}\right]\hat{x}$$

$$\boxed{\vec{E} = \left[\frac{\rho x}{\epsilon_0} - \frac{V_0}{L} - \frac{\rho L}{2\epsilon_0}\right]\hat{x}}$$

(2) $\rho = \frac{-\rho_0 x}{L}$ & $E=0,\ x=0$

$V=0,\ x=L$



$$\nabla^2 V = -\rho/\epsilon_0 \Rightarrow \frac{\partial^2 V}{\partial x^2} = \frac{\rho_0 x}{L\epsilon_0} \Rightarrow \frac{\partial V}{\partial x} = \frac{\rho_0}{L\epsilon_0}\frac{x^2}{2} + A$$

$$\Rightarrow V = \frac{\rho_0}{2L\epsilon_0}\frac{x^3}{3} + Ax + B$$

$$\Rightarrow V = \frac{\rho_0 x^3}{6L\epsilon_0} + Ax + B \quad \text{—(1)}$$

Bcoz we have B.C. at $x=0$, $E=0$ so Ist find E.

$$E = -\nabla V = -\frac{\partial V}{\partial x} \Rightarrow E = \left(-\frac{\rho_0 x^2}{2\epsilon_0 L} - A\right)\hat{x}$$

At $x=0$, $E=0 \Rightarrow \boxed{A=0}$

Now $V=0$, $x=L \Rightarrow 0 = \frac{\rho_0 L^3}{6L\epsilon_0} + 0 + B$

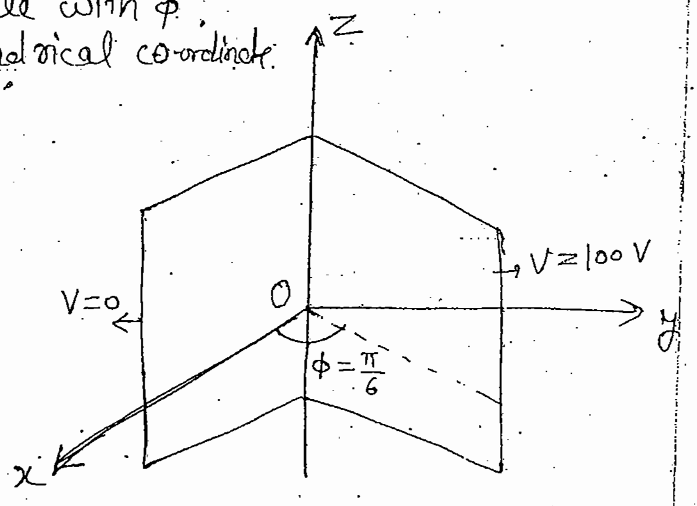
$$\Rightarrow \boxed{B = -\frac{\rho_0 L^2}{6\epsilon_0}}$$

So $\boxed{V = \frac{\rho_0 x^3}{6\epsilon_0 L} - \frac{\rho_0 L^2}{6\epsilon_0}}$

Now, $\vec{E} = -\vec{\nabla} V = -\frac{\partial V}{\partial x}$

$$\boxed{\vec{E} = \frac{-\rho_0 x^2}{2\epsilon_0 L}\hat{x}}$$

(3) $\nabla^2 V = 0$, V is variable with ϕ.
So take cylendrical co-ordinate.



$\Rightarrow \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0$

$\frac{\partial V}{\partial \phi} = A$

$\boxed{V = A\phi + B}$

At $\phi = 0$, $V=0 \Rightarrow$

$0 = 0 + B \Rightarrow \boxed{B=0}$

At $\phi = \frac{\pi}{6}$, $V = 100\,V$

$100 = A\frac{\pi}{6} \Rightarrow \boxed{A = \frac{600}{\pi}}$

So $\boxed{V = \frac{600\phi}{\pi}}$

So $\vec{E} = -\vec{\nabla} V = -\frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{\phi}$

$\vec{E} = -\frac{1}{\rho}\left[\frac{600}{\pi}\right]\hat{\phi}$

$$\boxed{\vec{E} = \frac{-600}{\rho\pi}\hat{\phi}}$$

Que :- The Electrostatic potential $V(x,y)$ in free space where $\rho = 0$, is given by $V(x,y) = 4e^{2x} + f(x) - 3y^2$. Given that x-component of electric field E_x and V are zero at origin. Find $f(x)$.

$\rho = 0$ So Use Laplace eqn.

$$\nabla^2 V = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$\frac{\partial^2}{\partial x^2}\left[4e^{2x} + f(x) - 3y^2\right] + \frac{\partial^2}{\partial y^2}\left[4e^{2x} + f(x) - 3y^2\right] = 0$$

$$\frac{\partial}{\partial x}\left[8e^{2x} + f'(x)\right] + \frac{\partial}{\partial y}\left[-6y\right] = 0$$

$$16e^{2x} + f''(x) - 6 = 0$$

$$\boxed{f''(x) = 6 - 16e^{2x}} \text{ ——— (1)}$$

or $\Rightarrow$ $\boxed{\frac{d^2 f(x)}{dx^2} = 6 - 16e^{2x}}$

Integrate $f'(x) = \frac{df(x)}{dx} = 6x - 8e^{2x} + C$ ——— (2)

Again $f(x) = 6\frac{x^2}{2} - 8\frac{e^{2x}}{2} + Cx + D$

$$f(x) = 3x^2 - 4e^{2x} + Cx + D \text{ ——— (3)}$$

$$\vec{E} = -\vec{\nabla} V = \underbrace{-\frac{\partial V}{\partial x}\hat{x}}_{x\text{-comp.}} - \frac{\partial V}{\partial y}\hat{y}$$

So $E_x = -\frac{\partial V}{\partial x}\hat{x}$ $\Rightarrow$ $E_x = -8e^{2x} - f'(x)$

At $x=0$, $y=0$, $E_x = 0$ $\Rightarrow$ $0 = -8 - f'(0)$

$\Rightarrow$ $\boxed{f'(0) = -8}$ ——— (4)

eqn (2) $\Rightarrow$ $f'(0) = 6(0) - 8 + C$ $\Rightarrow$ $-8 = 0 - 8 + C$

$\Rightarrow$ $\boxed{C = 0}$ ——— (5)

At $x=0$, $y=0$, $V=0$ $\Rightarrow$ $0 = 4 + f(0) - 0$

$\Rightarrow$ $\boxed{f(0) = -4}$ ——— (6)

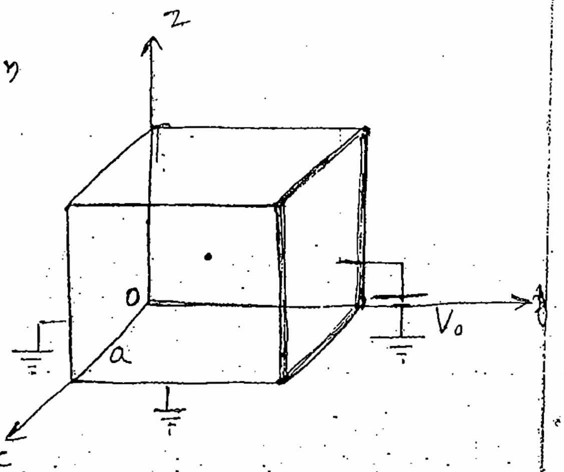
eqn (3) $\Rightarrow$ $f(0) = 0 - 4 + 0 + D$ $\Rightarrow$ $-4 = -4 + D$

$\Rightarrow$ $\boxed{D = 0}$ ——— (7)

Putting the value of C & D into (3),

$$\boxed{f(x) = 3x^2 - 4e^{2x}} \quad \text{—(8)}$$

Qu :- The Cube shown in the figure have 5 sides grounded & 6th side is insulated from the others & is hold at a constant potⁿ V_0. What is the potⁿ at the centre of the cube.



Here We need to solve Laplace eqⁿ in 3 dim. But this is difficult.

So Directely, Learn

$\boxed{\text{Pot}^n \text{ at centre is the average of all the faces.}}$

$$V_{centre} = \frac{V_1 + V_2 + V_3 + V_4 + V_5 + V_6}{6}$$

$V_1 = V_2 = V_3 = V_4 = V_5 = 0$ & $V_6 = V_0$

So $\boxed{V_{centre} = \frac{V_0}{6}}$

⇒ The surfaces of the cube is at the potⁿ V_0. What is the potⁿ at the centre?

then $V_{centre} = \frac{V_0 + V_0 + V_0 + V_0 + V_0 + V_0}{6}$

$V_{centre} = \frac{6V_0}{6} \Rightarrow \boxed{V_{centre} = V_0}$

If 2 surfaces are insulated & all other are grounded then $V_{centre} = \frac{V_0 + V_0 + 0 + 0 + 0 + 0}{6} = \frac{2V_0}{6}$

$\boxed{V_{centre} = \frac{V_0}{3}}$

We can apply this Rule on any Regular surface havi[ng] no. of faces more than 6 (or less than 6 if regular)

Multipole Expansion :-

Monopole means = total charge

for Monopole , $V \propto \frac{1}{r}$, $E \propto \frac{1}{r^2}$

dipole , $V \propto \frac{1}{r^2}$, $E \propto \frac{1}{r^3}$

Qudropole , $V \propto \frac{1}{r^3}$, $E \propto \frac{1}{r^4}$

Octopole , $V \propto \frac{1}{r^4}$, $E \propto \frac{1}{r^5}$

If Monople mom is Non zero then Monopole is dominting.

If total charge = 0 then monopole = 0 so dipole is dominating.

If dipole = 0 then Qudroupole is dominating. & so on.

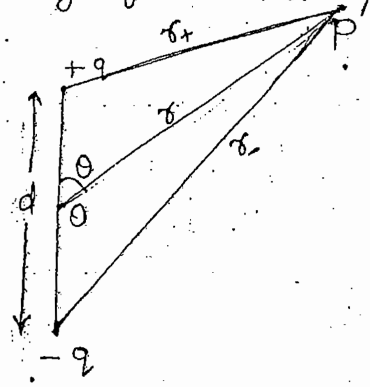
Electric field due to $\oint E \cdot dS = \frac{q}{\epsilon_0}$ is due to monople.

If it gives $E = 0$. It means elec. field of monopole is zero. May be E due to dipole is Non-zero.

Question :- A Dipole consist of 2 equal & opposite charges $\pm q$ seperated by a ~~small~~ distance d. Find the approximate pot^n at points far away from the dipole

⇒ Line joining the charges called :

Axis of the dipole. & It is always along z-dir^n.



Pot^n at P, due to $+q$ & $-q$ are

$$V_+ = \frac{q}{4\pi\epsilon_0 r_+} \quad \text{———} (1)$$

$$V_- = -\frac{1}{4\pi\epsilon_0}\frac{q}{r_-} \quad \text{———} (2)$$

$\boxed{r >> d}$

This cond^n should be satisfied. Only then we get seperate value of V_+ & V_-.

By cosine law,

$$r_+^2 = r^2 + \frac{d^2}{4} - rd\cos\theta$$

$$r_-^2 = r^2 + \frac{d^2}{4} + rd\cos\theta$$

Resultant Pot^n ⇒ $V = V_+ + V_-$

$$V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_+} + \frac{1}{r_-}\right]$$

$$V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\left(r^2+\frac{d^2}{4}-rd\cos\theta\right)^{1/2}} - \frac{1}{\left(r^2+\frac{d^2}{4}+rd\cos\theta\right)^{1/2}}\right]$$

$$= \frac{q}{4\pi\epsilon_0 r}\left[\left(1+\frac{d^2}{4r^2}-\frac{d}{r}\cos\theta\right)^{-1/2} - \left(1+\frac{d^2}{4r^2}+\frac{d}{r}\cos\theta\right)^{-1/2}\right]$$

$$= \frac{q}{4\pi\epsilon_0 r}\left[\left\{1-\frac{1}{2}\left(\frac{d^2}{4r^2}-\frac{d}{r}\cos\theta\right)\right\} - \left\{1-\frac{1}{2}\left(\frac{d^2}{4r^2}+\frac{d}{r}\cos\theta\right)\right\}\right]$$

$$= \frac{q}{4\pi\epsilon_0 r}\left[\left(1-\frac{d^2}{8r^2}+\frac{d}{2r}\cos\theta\right) - \left(1-\frac{d^2}{8r^2}-\frac{d}{2r}\cos\theta\right)\right]$$

$$= \frac{q}{4\pi\epsilon_0 r}\left[\left(1-\frac{d}{2r}\cos\theta\right) - \left(1-\frac{d}{2r}\cos\theta\right)\right]$$

{ $r >> d$ so neglect $\frac{d^2}{8r^2}$ ~~cos~~ }

$$= \frac{q}{4\pi\epsilon_0 r}\,\frac{2d\cos\theta}{2r}$$

$$\boxed{V = \frac{q}{4\pi\epsilon_0}\,\frac{d\cos\theta}{r^2}}$$ at large distance.

Now, Dipole Moment

$$\vec{p} = q\vec{d}$$

Dir^n of $\vec{p}$ will be from $-q$ to $+q$. $\vec{p}$ is along z-axis

so $\vec{p} = qd\,\hat{z}$

so $$\boxed{V = \frac{1}{4\pi\epsilon_0}\,\frac{\vec{p}\cdot\hat{r}}{r^2}}$$

{ $\vec{p}\cdot\hat{r} = qd\,\hat{z}\cdot\hat{r}$
$= qd\,\underset{\sim}{zr}\cos\theta$
$= qd\cos\theta$
$\hat{z}$ & $\hat{r}$ $\rightarrow$ unit vector

This is the pot^n at P. Its

$\boxed{\text{Monopole Moment} \Rightarrow +q-q=0}$

$$V_{\substack{axial\\(\theta=0^\circ)}} = \frac{qd}{4\pi\epsilon_0 r^2} = \frac{p}{4\pi\epsilon_0 r^2}$$

$V_{\substack{equitorial\\(\theta=90^\circ)}} = 0$ { along equitorial plane angle is 90° }

Pot^n is maximum along axial dir^n.

Electric field :- $\vec{E} = -\vec{\nabla}V$

$$\vec{E} = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta}$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{-2d\cos\theta}{r^3}\hat{r} - \frac{1}{r}\frac{d}{r^2}(-\sin\theta)\hat{\theta}\right]$$

$$= \frac{qd}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

$$\boxed{\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]}$$

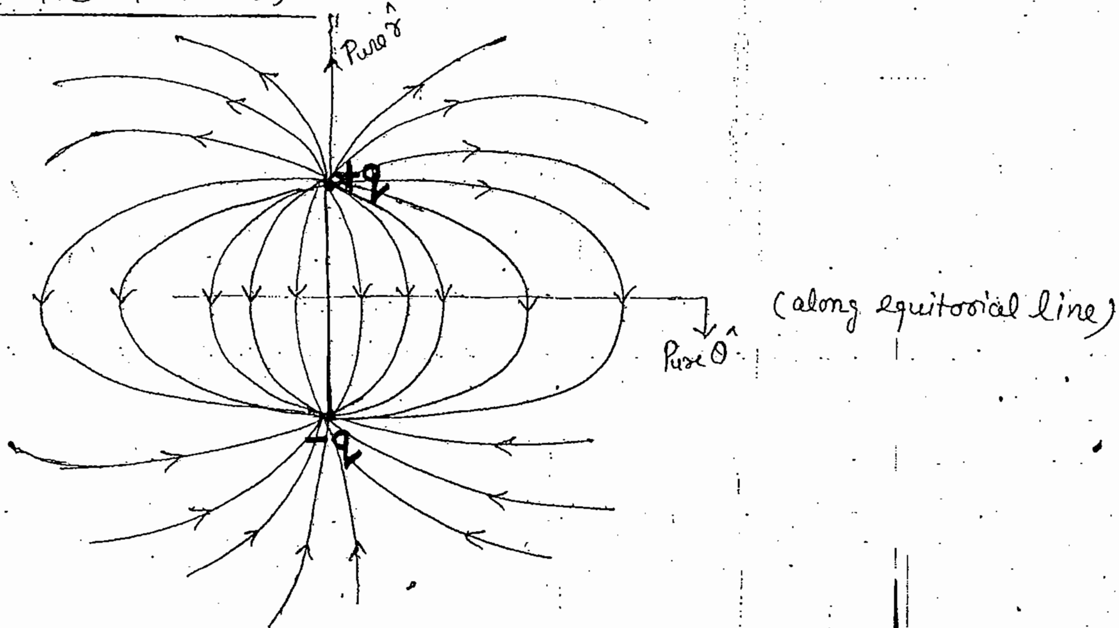
$$\vec{E}_{axial} = \frac{p}{4\pi\epsilon_0 r^3}(2) \Rightarrow \boxed{\vec{E}_{axial} = \frac{2p}{4\pi\epsilon_0 r^3}\hat{r}} \quad \{\theta = 0°\}$$

$$\boxed{\vec{E}_{equitorial} = \frac{p}{4\pi\epsilon_0 r^3}\hat{\theta}} \quad \{\theta = 90°\}$$

Magnitude of $\vec{E}$ is double at axial point as compare to equitorial point.

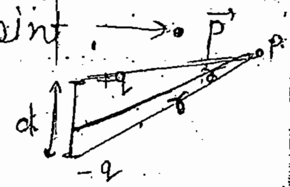
There are 2 dir^n of electric field [$\hat{r}$ & $\hat{\theta}$]

Electric field Lines :-



Point dipole :- dipole is concentrated at a point $\longrightarrow \cdot\, \vec{P}$

Physical dipole :- When $\boxed{d << r}$



Multipole Expansion of V :-

$$V = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\theta)\,\rho(r')\,d\tau'$$

$$dz' = r'^2 \sin\theta \, d\theta \, d\phi \, dr'$$

Where $P_n(\cos\theta) \longrightarrow$ Legendre's Polynomial.

for $n=0 \longrightarrow$ Monopole
for $n=1 \longrightarrow$ Dipole
$n=2 \longrightarrow$ Quadrapole

For $[n=0,\ P_0(x)=1]$ & $[n=1,\ P_1(x)=x]$

$$V = \underbrace{\frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(r')\,d\tau'}_{\text{Monopole term}} + \underbrace{\frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\theta\,\rho(r')\,d\tau'}_{\text{Dipole term}}$$

Dipole Mom.

$$\boxed{\begin{aligned}\vec{p} &= \int r'\cos\theta\,\rho(r')\,d\tau'\,\hat{z} \\ \vec{p} &= \int z\,\rho(r')\,d\tau'\,\hat{z}\end{aligned}} \longrightarrow \text{for Volume charge}$$

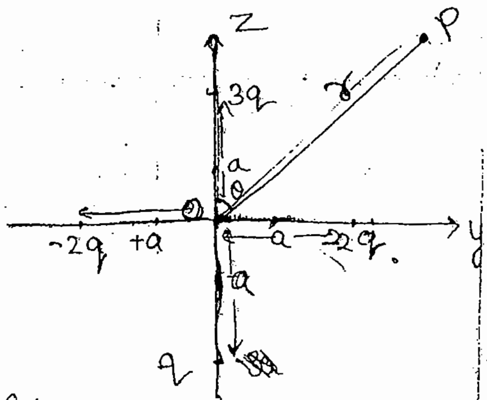
for Surface charge $\longrightarrow \boxed{\vec{p} = \int z\,\sigma(r')\,da'\,\hat{z}}$

for line charge $\longrightarrow \boxed{\vec{p} = \int z\,\lambda\,dl\,\hat{z}}$

for point charge $\longrightarrow \boxed{\vec{p} = q\,d}$

1 * If monopole moment (Total charge) is zero then dipole moment is independent of choice of Origin otherwise dipole moment depends upon choice of origin.

Que :- Find the dipole moment of this configuration. Also find approximate potn at far distance r.



Monopole moment =
Total charge $\boxed{Q=0}$

Origin is at O. & We have to find potn at P.

Monopole mom. = 0 so $V \propto \frac{1}{r} \Rightarrow 0$ [dipole mom. is dominating]
i.e. dipole mom. is independent of choice of origin

Dipole Mom. ⇒ $\vec{p} = 3qa\hat{z} + q(-a)\hat{z} + (-2q)(a)\hat{y}$
$+ (-2q)(+a)\hat{y}$

$$\boxed{\vec{p} = 2qa\hat{z}}$$

Potential ⇒ $V = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\frac{2qa\cos\theta}{r^2}}$$

It is approximat pot^n bcoz we are neglecting other ~~terms (Q.M.~~

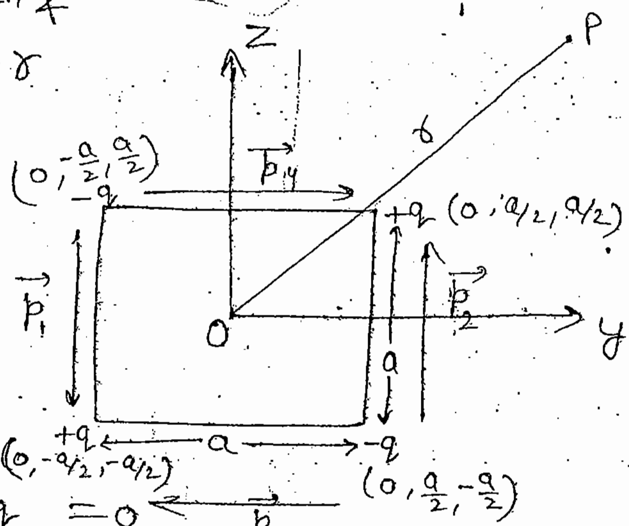
Que: ~~find~~ Dominent term of pot^n & electric field at large distance r will varry as

(a) $V \propto \frac{1}{r^2}$ & $E \propto \frac{1}{r^3}$

(b) $V \propto \frac{1}{r}$, $E \propto \frac{1}{r^2}$

✓ (c) $V \propto \frac{1}{r^3}$, $E \propto \frac{1}{r^4}$

(d) $V \propto \frac{1}{r^4}$, $E \propto \frac{1}{r^5}$



Monopole Mom $= +q - q + q - q = 0$

Dipole Mom. $= +q\frac{a}{2}\hat{y} + q\frac{a}{2}\hat{z} + (-q)\frac{a}{2}\hat{y} + (-q)\left(-\frac{a}{2}\right)\hat{z}$
$+ q\left(-\frac{a}{2}\right)\hat{y} + q\left(-\frac{a}{2}\right)\hat{z} + (-q)\left(-\frac{a}{2}\right)\hat{y} + (-q)\left(\frac{a}{2}\right)\hat{z}$

$= \cancel{\frac{qa}{2}}\hat{y} + \cancel{\frac{qa}{2}}\hat{z} - \cancel{\frac{qa}{2}}\hat{y} + \cancel{\frac{qa}{2}}\hat{z} - \cancel{\frac{qa}{2}}\hat{y} - \cancel{\frac{qa}{2}}\hat{z} + \cancel{\frac{qa}{2}}\hat{y} - \cancel{\frac{qa}{2}}\hat{z}$
$= 0$

~~di~~ There are 4 dipoles. Dipole p_1 & p_2 are equal & opposite. Also Dipole p_3 & p_4 are equal & opposite. So Total Dipole Moment $\boxed{\vec{p} = 0}$

Note :- This is a special case. But In every case we can not judge dipole mom. directely. We have to find it by formula.

Qu. Find Dipole Mom.

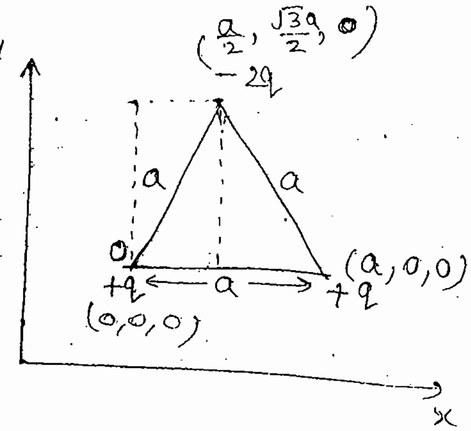
Dipole Mom. $= +q \times 0$

$+q(a)\hat{x}$

$+(-2q)\frac{a}{2}\hat{x}$

$+(-2q)\left(\frac{\sqrt{3}a}{2}\right)\hat{y}$

$$\boxed{\vec{p} = -\sqrt{3}\, q\, a\, \hat{y}}$$



Qu :- Two point charges $3q$ & $-q$ are seperated by distance a, for each of the arrangment. Find.

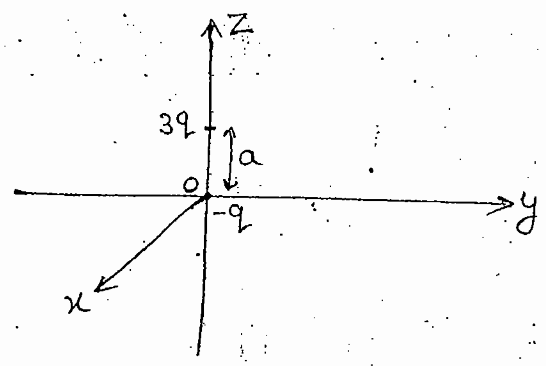
(i) Monopole Mom.

(ii) Dipole Mom.

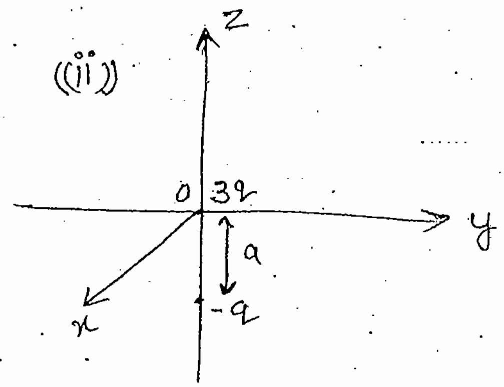
(iii) Approximate potⁿ at large r

(iv) Electric field

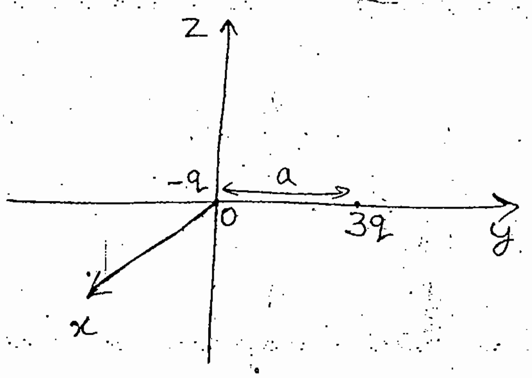
(i)



(ii)



(iii)



for (i) Monopole Mom. $= 3q - q = 2q$

Dipole mom. $= 3q(a)\hat{z} + (-q)(0)\hat{y} = 3qa\hat{z}$

Potⁿ $V = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} \Rightarrow V = \frac{1}{4\pi\epsilon_0}\frac{3qa\cos\theta}{r^2}$ (dipole term)

~~Electric field $\vec{E} = -\vec{\nabla}V$~~

$V = \frac{1}{4\pi\epsilon_0}\frac{2q}{r}$ (monopole term)

Total $V = \frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\cos\theta}{r^2}\right]$

Electric field $\vec{E} = -\vec{\nabla} V$

$$\vec{E} = \frac{-q}{4\pi\epsilon_0}\left[-\frac{2}{r^2}\hat{r} + 3a\cos\theta\left(\frac{-2}{r^3}\right)\hat{r} + \frac{1}{r}\frac{3a}{r^2}(-\sin\theta)\,\hat{\theta}\right]$$

$$= \frac{-q}{4\pi\epsilon_0}\left[-\left(\frac{2}{r^2} + \frac{6a\cos\theta}{r^3}\right)\hat{r} - \frac{3a}{r^3}\sin\theta\,\hat{\theta}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\left(\frac{2}{r^2} + \frac{6a\cos\theta}{r^3}\right)\hat{r} + \frac{3a}{r^3}\sin\theta\,\hat{\theta}\right]$$

(ii) Monopole mom. $= 2q$

Dipole mom. $= (-q)(-a)\hat{z} + 3q(0)\hat{y}$

$\vec{p} = qa\hat{z}$

$$V = \frac{1}{4\pi\epsilon_0}\left[\cancel{2q}\ \frac{2q}{r} + \frac{qa\cos\theta}{r^2}\right]$$

$$\vec{E} = -\vec{\nabla} V$$

$$= -\frac{q}{4\pi\epsilon_0}\left[-\frac{2}{r^2}\hat{r} + a\cos\theta\left(\frac{-2}{r^3}\right)\hat{r} + \frac{1}{r}\frac{a}{r^2}(-\sin\theta)\hat{\theta}\right]$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\left[\left(\frac{2}{r^2} + 2\frac{a\cos\theta}{r^3}\right)\hat{r} + \frac{a}{r^3}\sin\theta\,\hat{\theta}\right]$$

(iii) Monopole mom $= 2q$

Dipole Mom. $= 3q(a)\hat{y} + (-q)(0)\hat{z} = 3qa\hat{y}$

$\vec{p} = 3qa\hat{y}$

$$V = \frac{1}{4\pi\epsilon_0}\frac{2q}{r} + \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{2q}{r} + \frac{1}{4\pi\epsilon_0}\frac{3qa\,\hat{y}\cdot\hat{r}}{r^2}$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\sin\theta\sin\phi}{r^2}\right]$$

$$\vec{E} = -\vec{\nabla} V$$

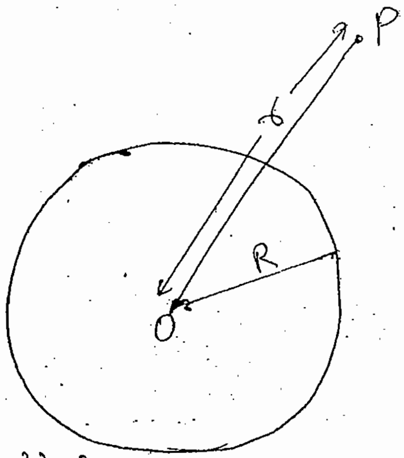
$$= \frac{-q}{4\pi\epsilon_0}\left[-\frac{2}{r^2}\hat{r} + 3a\sin\theta\sin\phi\left(\frac{-2}{r^3}\right)\hat{r} + \frac{1}{r}\frac{3a\sin\phi(\cos\theta)}{r^2}\hat{\theta} + \frac{3a\sin\theta}{r\sin\theta\, r^2}(\cos\phi)\hat{\phi}\right]$$

$$= +\frac{q}{4\pi\epsilon_0}\left[\left(\frac{2}{r^2} + \frac{6a\sin\theta\sin\phi}{r^3}\right)\hat{r} - \frac{3a\sin\phi\cos\theta}{r^3}\hat{\theta} - \frac{3a\cos\phi}{r^3}\hat{\phi}\right]$$

Ques :- A sphere of radius R centred at the origin & carries a charge density $\rho(r,\theta) = \rho_0 (a^2 - r^2) \cos\theta$. The electric field at a distance r from the centre of the sphere will varry as

(a) $E \propto \frac{1}{r^2}$ ✓(b) $E \propto \frac{1}{r^3}$

(c) $E \propto \frac{1}{r^4}$ (d) $E \propto \frac{1}{r^4}$



<u>Monopole Mom</u> :-

Total charge

$$Q = \int \rho \, d\tau$$

$$= \int_0^R \int_0^\pi \int_0^{2\pi} \rho_0 (a^2 - r^2) \cos\theta \, r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \rho_0 (2\pi) \underbrace{\int_0^\pi \sin\theta \cos\theta \, d\theta}_{0} \int_0^R (a^2 - r^2) r^2 \, dr$$

$$\boxed{Q = 0}$$

<u>Dipole Mom.</u> :-

$$\vec{p} = \int r' \cos\theta \, \rho(r') \, d\tau' \, \hat{z}$$

$$= \int_0^R \int_0^\pi \int_0^{2\pi} r \cos\theta \, \rho_0 (a^2 - r^2) \cos\theta \, r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= 2\pi \rho_0 \int_0^\pi \cos^2\theta \sin\theta \, d\theta \int_0^R r (a^2 - r^2) r^2 \, dr$$

$$\cancel{\Rightarrow 2\pi \rho_0 \int_0^\pi (1 - \sin^2\theta) \sin\theta \, d\theta \int_0^R (a^2 r^3 - r^5) \, dr}$$

$$\cancel{\Rightarrow 2\pi \rho_0 \left[-\cos\theta - 3 \sin\theta \cos\theta \right]_0^\pi}$$

$$\int_0^\pi \cos^2\theta \sin\theta \, dt = -\int t^2 \, dt$$

Let $\cos\theta = t$

$-\sin\theta \, d\theta = dt$

$$= -\frac{t^3}{3}$$

$$= -\left[\frac{(\cos\theta)^3}{3} \right]_0^\pi = -\frac{1}{3} \left((-1)^3 - 1 \right)$$

$$= -\frac{1}{3} (-2) = +\frac{2}{3}$$

$$\int_0^R (a^2 r^3 - r^5)\, dr = \left[a^2 \frac{r^4}{4} - \frac{r^6}{6}\right]_0^R = a^2 \frac{R^4}{4} - \frac{R^6}{6}$$

So $\vec{p} = 2\pi \rho_0 \left(\frac{2}{3}\right)\left(a^2 \frac{R^4}{4} - \frac{R^6}{6}\right)$

$$\boxed{\vec{p} = \frac{4\pi}{3} \rho_0 \left(\frac{R^4 a^2}{4} - \frac{R^6}{6}\right)}$$

* If dipole mom = 0
then

Qudroopole Mom $= \int r^2 \frac{(3\cos^2\theta - 1)}{2} \rho\, d\tau$

Que :- $\rho = \frac{KR}{r^2} (R - 2r) \sin\theta$ [In previous Question]

Monopole Mom

$$Q = \int \rho\, d\tau = \int_0^R \int_0^\pi \int_0^{2\pi} \frac{KR}{\cancel{r^2}} (R - 2r) \sin\theta \; \cancel{r^2} \sin\theta\, d\theta\, d\phi$$

$$= KR \int_0^R (R - 2r) \int_0^\pi \sin^2\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= KR(2\pi)\left[-\frac{\cancel{2} r^2}{\cancel{2}} R\right]_0^R \int_0^\pi \frac{1 - \cos 2\theta}{2}\, d\theta$$

$$= 2\pi KR(-R^2) \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^\pi$$

$$= -\pi R^3 K [\pi - 0] = -\pi^2 R^3 K$$

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6/8/2012

Dielectrics :- Dielectrics are the perfect Insulators & they have the property of polarization. If we apply electric field to a dielectric, it will be polarised.

Dielectrics are mainly of 2 types :-

1) Polar Dielectrics
2) Nonpolar Dielectrics

Polar dielectrics are those dielectrics which are made up of polar molecules & Non-polar dielectrics are those which made up of nonpolar molecules.

Polar Molecules - Those have permanent dipole moment
more precisely - permanent electric dipole moment.

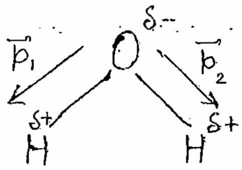
{ electric dipole Moment → dipole moment
{ Magnetic " " → magnetic moment }

Non-polar Molecule - Those don't have permanent dipole moment.

Ex :-

Polar Molecules	Non Polar molecules
NaCl	N_2
HCl	CO_2
H_2O	H_2
	O_2

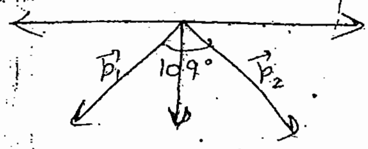
(1)



O → oxygen is highly electronegative
O share 1-1 e^- with H to ~~com~~ achieve stability & make covalent bond.

Magnitude of dipole mom. depends upon charge & distance

If we resolve these two dipole mom. in components then It has a permanent dipole Moment.



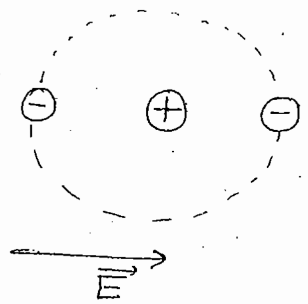
(2) N ..... N both have same electronegativity
So shared e^- pair is b/w N and N
So Net dipole Mom = 0

(3) $O^{\delta-}$—$C^{\delta+}$—$O^{\delta-}$ both dipole mom. are in opposite dir^n So
Net dipole mom = 0

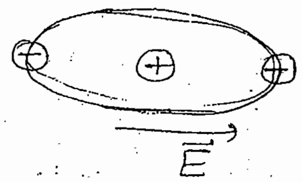
Homonuclear Molecules are Non-polar.
Hetro nuclear

Induced Dipole Moment :-

Net dipole mom. of an atom is 0 bcoz centre of −ve charges & +ve charges conside.



If we apply electric field then shape will change & e^- will shift from their position while nucleus will be fix. Now centre of +ve & −ve charge will not conside. It will accuire some dipole Mom.



This dipole mom. is not permanent dipole mom. It is called Induced dipole mom.

And $\vec{p}_{in} \propto \vec{E}$

$$\boxed{\vec{p}_{in} = \alpha \vec{E}_{loc}}$$

α is polarisability of the atom.

Polarisability :- Abality of polarisation is called polarisability.

This electric field is local electric field.

$$\vec{E} = \vec{E}_{local}$$

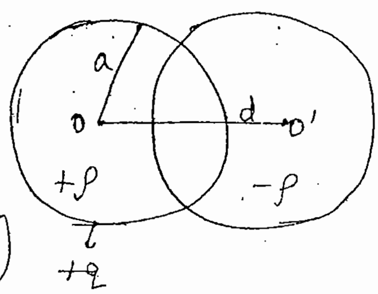
Que 1 :- A primitive model of an atom consist of a point charge $+q$ surrounded by a uniformly charge spherical cloud $(-q)$ of radius a. Calculate the atomic polarisability of the atom.

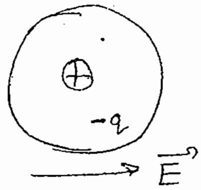
$$\vec{E} = \frac{\rho \vec{d}}{3\epsilon_0}$$

convert it in form of q.

$$\vec{E} = \frac{q\vec{d}}{4\pi\epsilon_0 a^3}$$

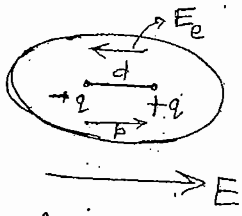
$$\left(\begin{aligned} q &= \rho V \\ \rho &= \frac{q}{V} = \frac{q}{\frac{4}{3}\pi a^3} \end{aligned} \right)$$





When we apply electric field then −ve charge displaced in the opposite dirn of $\vec{E}$

The distortion of shape of sphere is very small. The order of d is very small (less than Å)



The shape will change untill External ele. field E becomes equal to the internal equilibrium elec. field E_o.

$$\boxed{E = E_e = \frac{qd}{4\pi\epsilon_0 a^3}} \quad \text{——(1)}$$

Dipole mom.

$$p = qd$$

Dirn of $\vec{p}$ will be in the dirn of externally aplied elec. field. So dipole mom. accurie by the atom is

Induced dipole mom. i.e. $\boxed{p = 4\pi\epsilon_0 a^3 E}$ ——(2)

We have $p_{in} = \alpha E$ ——(3)

On Comparing, We get $\boxed{\alpha = 4\pi\epsilon_0 a^3}$ or $\boxed{\alpha = 3\epsilon_0 V}$

Polarisability. ——(4) ——(5)

V → volume of the atom

In C.G.S. Unit :- $\epsilon_0 = \frac{1}{4\pi}$

So $\alpha = a^3$, $\boxed{\text{unit} = cm^3}$ (volume)

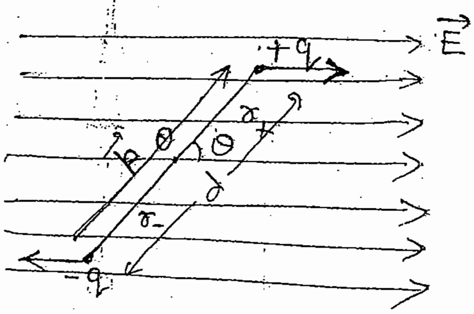
In M.K.S. Unit :-

$$\frac{N}{C} = \frac{1}{\epsilon_0} \times \frac{C}{m^2} \Rightarrow \epsilon_0 = \frac{C^2}{N\text{-}m^2}$$

Hence $\boxed{\text{Unit of } \alpha = \frac{C^2\text{-}m}{N}}$

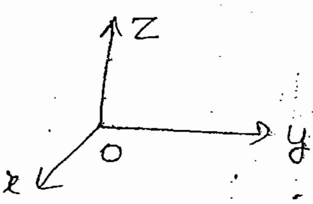
• Force & Torque on a permanent dipole in uniform Electric field :-

A dipole is placed in uniform electric field.



Electric field is uniform i.e. does not depand on space part.

$$\vec{E} = E_0 \hat{y}$$



force on the dipole,

$$\vec{F}_+ = q\vec{E} \quad ; \quad \vec{F}_- = -q\vec{E}$$

Total force on dipole $\vec{F} = \vec{F}_+ + \vec{F}_-$

$$\boxed{\vec{F} = 0}$$

In uniform electric field, force on dipole is zero.

<u>Note</u> :- If elec. field depend on space part → Non uniform then dipole will experience some force.

for a ideal dipole (have equal & opposite charges), ~~so~~ the point of rotation will be centre of mass O. Force due to rotation is Torque :-

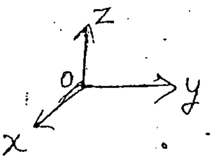
$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\vec{\tau} = \vec{r}_+ \times \vec{F}_+ + \vec{r}_- \times \vec{F}_-$$

$$\vec{\tau} = \frac{\vec{d}}{2} \times (q\vec{E}) + \left(-\frac{\vec{d}}{2}\right) \times (-q\vec{E})$$

$$\vec{\tau} = q(\vec{d} \times \vec{E})$$

$$\boxed{\vec{\tau} = \vec{p} \times \vec{E}}$$



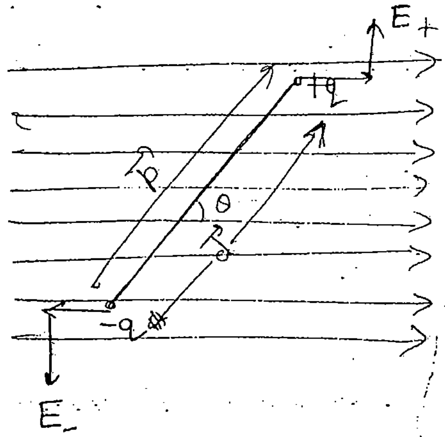
Dir^n of Torque → $(-\hat{x})$

Force & Torque in Non-Uniform Electric field :-

E_+ is different from E_-.

$$\vec{F}_{net} = q\vec{E}_+ - q\vec{E}_-$$

$$= q(\vec{E}_+ - \vec{E}_-)$$



d is very small so change in elec. field is not drastic (small change)

So $\boxed{\vec{F}_{net} = q\,\Delta\vec{E}}$

$$\rightarrow \Delta\vec{E} = (\vec{d}\cdot\vec{\nabla})\vec{E}$$

So $\vec{F}_{net} = q(\vec{d}\cdot\vec{\nabla})\vec{E}$

$$\boxed{\vec{F}_{net} = (\vec{p}\cdot\vec{\nabla})\vec{E}}$$ — Imp

If $\vec{F}_{net} = \vec{\nabla}(\vec{p}\cdot\vec{E})$ → This is conditionally True only, if p is not a fun^n of position.

from Gradient theorem

$$dA = (\vec{\nabla}A)\cdot d\vec{r}$$

dA ↓ change in A

$$\vec{\nabla}A = \frac{\partial}{\partial x}A\hat{i} + \frac{\partial}{\partial y}A\hat{j} + \frac{\partial}{\partial z}A\hat{k}$$

$$d\vec{A} = (d\vec{r}\cdot\vec{\nabla})\vec{A}$$

$$\vec{\nabla}(\vec{p}\cdot\vec{E}) = \vec{p}\times\underbrace{(\vec{\nabla}\times\vec{E})}_{=0} + \vec{E}\times\underbrace{(\vec{\nabla}\times\vec{p})}_{=0} + (\vec{p}\cdot\vec{\nabla})\vec{E} + \underbrace{(\vec{E}\cdot\vec{\nabla})\vec{p}}_{=0}$$

In electrostatic field   p is not func^n of position → ″

So $\boxed{\vec{\nabla}(\vec{p}\cdot\vec{E}) = (\vec{p}\cdot\vec{\nabla})\vec{E}}$

In Non Uniform elec. field both force & torque are Non-zero

<u>Torque</u> :- $\vec{\tau} = \underbrace{(\vec{p}\times\vec{E})}_{\text{uniform}} + \underbrace{(\vec{r}\times\vec{F})}_{\text{Non-uniform}}$

where r → distance from axis of rotation.

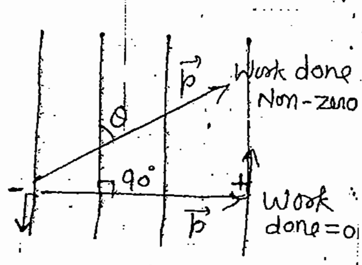
$$\boxed{\vec{\tau} = (\vec{p}\times\vec{E}) + \vec{r}\times[(\vec{p}\cdot\vec{\nabla})\vec{E}]}$$

For a Pure (point or perfect) dipole $\boxed{\vec{\tau} = \vec{p}\times\vec{E}}$ for uniform & Non-uniform elec. field both.

<u>Potential Energy of a dipole</u> :-

Work done under torque τ, will store in form of pot^n energy.

$$dU = dW = \tau\, d\theta$$
$$= |\vec{p}\times\vec{E}|\, d\theta$$
$$= pE\sin\theta\, d\theta$$



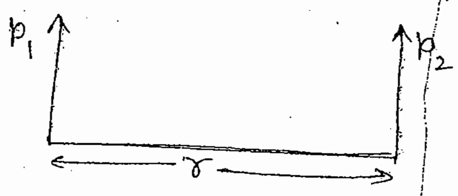
$$U = W = \int_{90°}^{\theta} pE\sin\theta\, d\theta = pE(-\cos\theta)_{90}^{\theta}$$

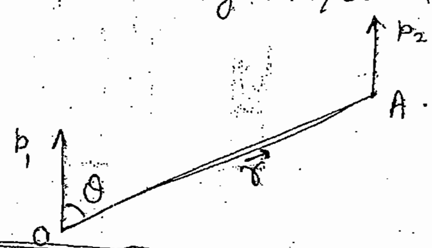
$U = -pE\cos\theta \Rightarrow \boxed{U = -\vec{p}\cdot\vec{E}}$

-ve means work done itself.

Minimum pot^n energy signifies the stability.
If p & E are in same dir^n then $\theta = 0$

<u>Interaction Energy of 2 Dipoles</u> :- depends upon both the distance b/w the dipoles r as well as angle b/w them.





$$\boxed{U = \frac{1}{4\pi\epsilon_0 r^3}\left[\vec{p}_1\cdot\vec{p}_2 - 3(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})\right]}$$

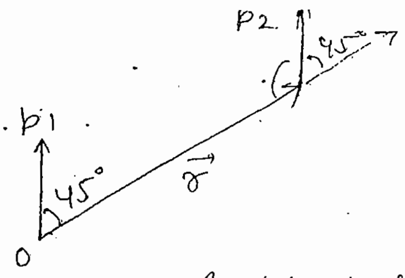
Ques :- find U ?

$$U = \frac{1}{4\pi\epsilon_0 r^3}\left[\vec{p}_1\cdot\vec{p}_2 - 3(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})\right]$$

$$= \frac{1}{4\pi\epsilon_0 r^3}\left[p_1p_2 - 3\,p_1\cos 45 \cdot p_2\cos 45\right]$$

$$= \frac{1}{4\pi\epsilon_0 r^3}\, p_1 p_2 \left(1 - \frac{3}{2}\right)$$

$$\boxed{U = \frac{-1}{4\pi\epsilon_0 r^3}\,\frac{p_1p_2}{2}}$$



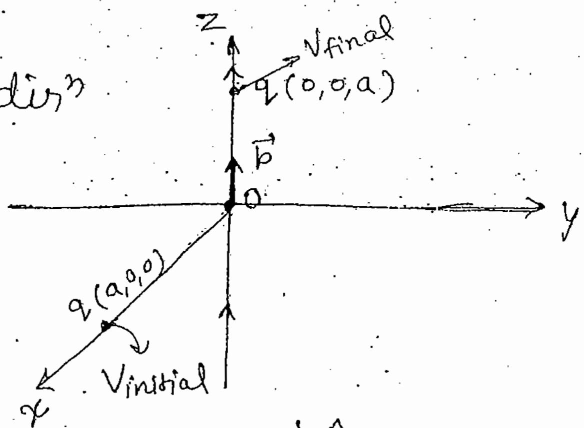
angle b/w p_1 & p_2 = 0

Ques :- A dipole $\vec{p}$ is situated at origin & points in z-dirⁿ
(a) What is the force on point charge q at $(a,0,0)$
(b) " " " " " at $(0,0,a)$
(c) How much work done required to move q from $(a,0,0)$ to $(0,0,a)$

$$\vec{F} = q\,\vec{E}_{dipole}$$

Electric field of a dipole in z-dirⁿ

$$\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$



(a) for q at $(a,0,0) \rightarrow \theta = 90°$

$$\vec{E} = \frac{p}{4\pi\epsilon_0 a^3}\left[0 + \hat{\theta}\right] \qquad (r = a)$$

$$\boxed{\vec{E} = \frac{p}{4\pi\epsilon_0 a^3}\,\hat{\theta} = \frac{-p}{4\pi\epsilon_0 a^3}\,\hat{z}}$$

$\hat{\theta} = 0 + 0 - \sin\theta\,\hat{z}$
$\hat{\theta} = -\hat{z}$ $\{\theta = 90°\}$

So $$\boxed{\vec{F} = \frac{qp}{4\pi\epsilon_0 a^3}\,\hat{\theta} = \frac{pq}{4\pi\epsilon_0 a^3}(-\hat{z})}$$

(b) for, q at $(0,0,a) \rightarrow \theta = 0°$

$$\boxed{\begin{aligned}\vec{E} &= \frac{p}{4\pi\epsilon_0 r^3}\left[2\hat{r}\right] \\ \vec{F} &= \frac{2pq}{4\pi\epsilon_0 a^3}\,\hat{z}\end{aligned}}$$

$[\hat{r} = \hat{z}]$

(C) Work done

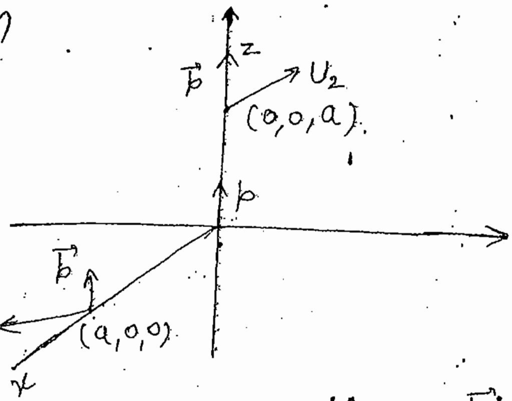
$$\Delta U = \Delta W = q\left(V_{final} - V_{initial}\right)$$

$$V_{final} = \frac{1}{4\pi\epsilon_0}\,\frac{p\cos\theta}{r^2} = \frac{1}{4\pi\epsilon_0}\,\frac{p}{a^2} \qquad (\theta = 0) \text{ at } (0,0,a)$$

$$V_{initial} = 0 \qquad (\theta = 90°) \text{ at } (a,0,0)$$

$$\boxed{W = \frac{1}{4\pi\epsilon_0}\,\frac{pq}{a^2}}$$

Without changing the orientation dipole is shifted from point $(a,0,0)$ to $(0,0,a)$



Angle b/w $\overset{(a,0,0)}{p_1}$ & $\overset{(0,0,0)}{p_2} = 0°$

$$\vec{p_1}\cdot\vec{p_2} = p^2$$

$$\vec{p_1}\cdot\hat{r} = p_i \cos 90° = 0$$

$$\vec{p_2}\cdot\hat{r} = p\cdot\cos 90° = 0$$

$$U = \frac{1}{4\pi\epsilon_0 r^3}\left[\vec{p_1}\cdot\vec{p_2} - 3(\vec{p_1}\cdot\hat{r})(\vec{p_2}\cdot\hat{r})\right]$$

$$U_1 = -\vec{p}\cdot\vec{E}$$
$$U_2 = -\vec{p}\cdot\vec{E}$$

$$\boxed{U_1 = \frac{p^2}{4\pi\epsilon_0 a^3}}$$

Angle b/w p_1 at $(0,0,0)$ & P_2 at $(0,0,a) = 0°$ so $\vec{p_1}\cdot\vec{p_2} = p^2$

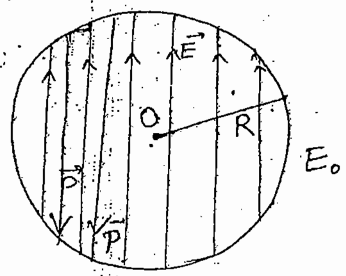
$$\vec{p_1}\cdot\hat{r} = p\hat{z} \quad \& \quad \vec{p_2}\cdot\hat{r} = p\hat{z}$$

$$U_2 = \frac{1}{4\pi\epsilon_0 a^3}\left[p^2 - 3p^2\right] \Rightarrow \boxed{U_2 = \frac{-2p^2}{4\pi\epsilon_0 a^3}}$$

- <u>Electric field of a uniformly polarised sphere</u> :-

$\vec{P}$ does not depend on $r, \theta, \phi, s, \phi, z$.

We have to find ele. field inside the sphere. $\boxed{E_{out} = 0}$



<u>Polarisation is dipole moment per unit volume</u>. This is a <u>Macroscopic</u> quantity & dipole mom. is <u>Microscopic</u>.

$$\vec{E} = \frac{\rho\vec{d}}{3\epsilon_0} = \frac{qd}{4\pi\epsilon_0 R^3} = \frac{\vec{p}}{3\times\frac{4}{3}\pi\epsilon_0 R^3}$$

$$\boxed{\vec{E} = -\frac{\vec{P}}{3\epsilon_0}}$$

Polarisation = dipole mom. per unit volume

Dir^n of $\vec{E}_{in}$ will be opposite to dir^n of $\vec{p}$ inside ~~i.e.~~ polarisation

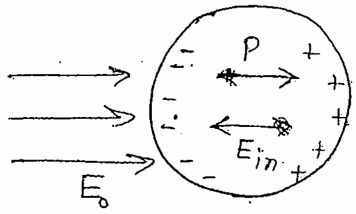
If $\vec{P}$ is uniform then $\vec{E}$ will be uniform.

- * $\vec{E}$ is uniform
- * opposite to polarisation
- * Magnitude is $\frac{P}{3\epsilon_0}$.

If we apply external ele. field E_0 then it will polarised the sphere then total elec. field will be less than E_0.

$$E_t < E_0$$

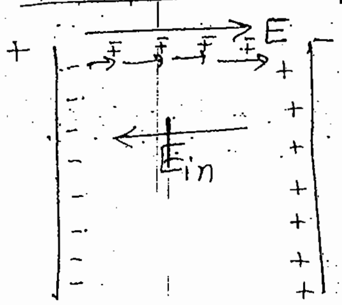
$$\boxed{\vec{E}_t = \vec{E}_0 - \frac{\vec{P}}{3\epsilon_0}}$$



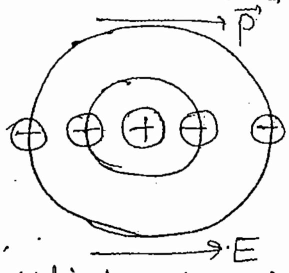
In free space polarisation $\vec{P}$ will be zero & in case of conducting sphere the internal elec. field will be equal to external ele. field.

Bound Charges :- bound charge is bound to the nuclei while free charge move freely.

Example :- Capacitor, filled with dielectric



When Elec. field is applied then polarisation will occur. & Dielectric will be polarised. & dipole is formed.



{ In solid there is a series of atoms

Here an internal ele. field will produce due to bound charges. This is in opposite dir^n of external applied $\vec{E}$. & E_{in} is less than E.

* Bound charges are of 2 types :-

→ Surface bound charges

→ Volume " "

⇒ if polarisation is uniform then ~~total effect of E.F.~~ net charge is only on surface then this is called Surface bound charge.

⇒ if polarisation is not uniform & Net charge is on surface & volume also then this is called Volume ~~ch~~ bound charge.

Surface bound charge density

$$\boxed{\sigma_b = \vec{P}\cdot\hat{n}}$$

{ vol. charge density → occurs only in non-uniform field } but surface for both