Origin of Quantum Mechanics

Acc. to classical Mechanics,
The particle can behave like a particle & wave behave like a wave. Particle show single character (behaviour).
Particle is repⁿ by E (energy) & mom. P.
Wave " " Max. wave eqⁿ.

$$\boxed{E = E_0 e^{i(k\cdot x - \omega t)}}$$

$$\boxed{B = B_0 e^{i(k\cdot x - \omega t)}}$$

To repⁿ the wave we must know the amp. & propogation vector.

amp. E_0, B_0

$$\boxed{K = \frac{2\pi}{\lambda}\hat{n}}$$

Intensity ~~or~~ energy incident per unit area per unit time.

Acc. to cla. mech. all the dynamical variables are ~~dis~~ continous.

If we know r then

$$v = \frac{dr}{dt}, \quad a = \frac{d^2r}{dt^2}, \quad F = ma, \quad F = -\nabla V$$
$$V = -\int F$$

$$P = mv, \Rightarrow K.E. = \frac{P^2}{2m}, \quad E = K.E. + P.E.$$

In classical mech. if we know position, we can calculate all the information

Particle & radiation ~~behaves~~ shows a single behaviour

Classical mech can't explain some phenomenon as photoelectric effect, compton effect, black body radiation. Energy exchange b/w radiation & matter take place continously.

Date :

Page No.

C.M. fails

↓

Q.M.

Old Q.M. | New Q.M.

Acc to new Q. Theory, particle is rep^n by a wave.
" " old " , Energy & ang. momentum both are quantised.

$$\boxed{E = n\hbar\omega}$$
$$\boxed{L = n\hbar}$$

Some phenomena explain by Old Q.M. not by C.M. :-

Particle Nature of radiation :-

(1) Photoelectric effect :- Here photon is a particle not a wave. Photon energy

$$\boxed{E = nh\nu}$$

Radiation comprise of particle.

Principle :-

The emission of e^- by incident radiation is called Photo-electric effect.

⇒ Exp. Observations :- ① If we incident radiation of less freq. than threshold freq. then no emission of electron.

$\nu < \nu_0$ there is no emission of e^-

② $\nu > \nu_0$ there is always emission of e^-

③ There is no time difference b/w incident & emission.

Time diff. $= 10^{-8}$ sec → negligible

i.e. e^- at once emitted.

④ The K.E. of emitted e^- depends only on freq. of incident radiation not on intensity.

These cond^{n}s should be satisfied.

If mag. field apply on e^- to accelerate it then force on it will be $\boxed{F = q(v \times B)}$

$W = F \cdot dl$ $\left[v = \frac{dl}{dt}\right]$

Work done by mag. force is zero.

If $W = 0$ then we can't change the K.E. of e^- so by applying mag. field, e^- can not be emitted.

Mag. forces only change the dir^n, no change in speed.

Now applying electric field, force on e^- is electrostatic force

$F = qE$

Work done is not zero, then K.E. can be change so e^- can be emitted.

Poor Intensity → less energy is supplied. Emission of e^- is possible but in more time (acc. to classical

Strong Intensity → More energy is supplied & this energy is used

$E = K.E._{max} + W$

$F(E) = 1 \quad E \leq E_F$, all the energy levels r filled.
$\quad\;\; = 0 \quad E > E_F$

Heighest filled level is the fermi level.

Work Fuc^n :- Mini energy required to move the highest occupied level to surface & remaining energy to supply the K.E.

In Photo electric effect, the e^- with max. K.E. is emitted.

If energy is $\geq$ Work fuc^n r, e^- emits.

* $\downarrow$ e^-'s K.E. depands on intensity.

$K.E_{max} = E - \phi W \Rightarrow K.E_{max} = \frac{hc}{\lambda} - \phi$

$\lambda_0 = \frac{hc}{\phi W} \Rightarrow \nu_0 = \frac{c}{\lambda_0} = \frac{W}{h} \Rightarrow W = h\nu_0$

so $\boxed{K.E_{max} = h(\nu - \nu_0)}$

Date :

Page No. :

If we include energy Quantisation law & particle nature of e^- then we can explain P.E. effect.

In strong intensity, no. of photons inci. / unit area / unit are large

Poor → " " " " small

Acc. to Q.M., E can be rep^n in terms of freq. ν

$E = h\nu$

$$\boxed{K.E_{max} = h(\nu - \nu_0)}$$

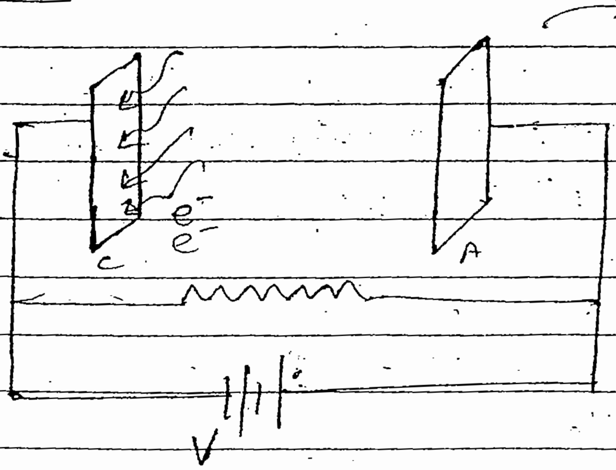
(1) If $\nu < \nu_0$, then $\nu - \nu_0 \to -ve$

so $K.E_{max} = -ve$

$K.E. = \frac{1}{2} m v^2$ ($+ve$) so K.E. is always +ve, but here K.E. is −ve. Not Possible i.e. no emission of e^-. So we have to supply more energy to move e^- upto the surface.

(2) If $\nu > \nu_0$:- $K.E_{max} = +ve$ then there is always emission of e^-. If this cond^n is satisfied then at once e^- emitted.

Stopping Pot^n :-



+ve pot^n w.r.to Ist electro then e^-s accumulte

−ve then e^-s repel.

e^- move from C → A

P.E. ↑ by eV_0 = ↓ in K.E

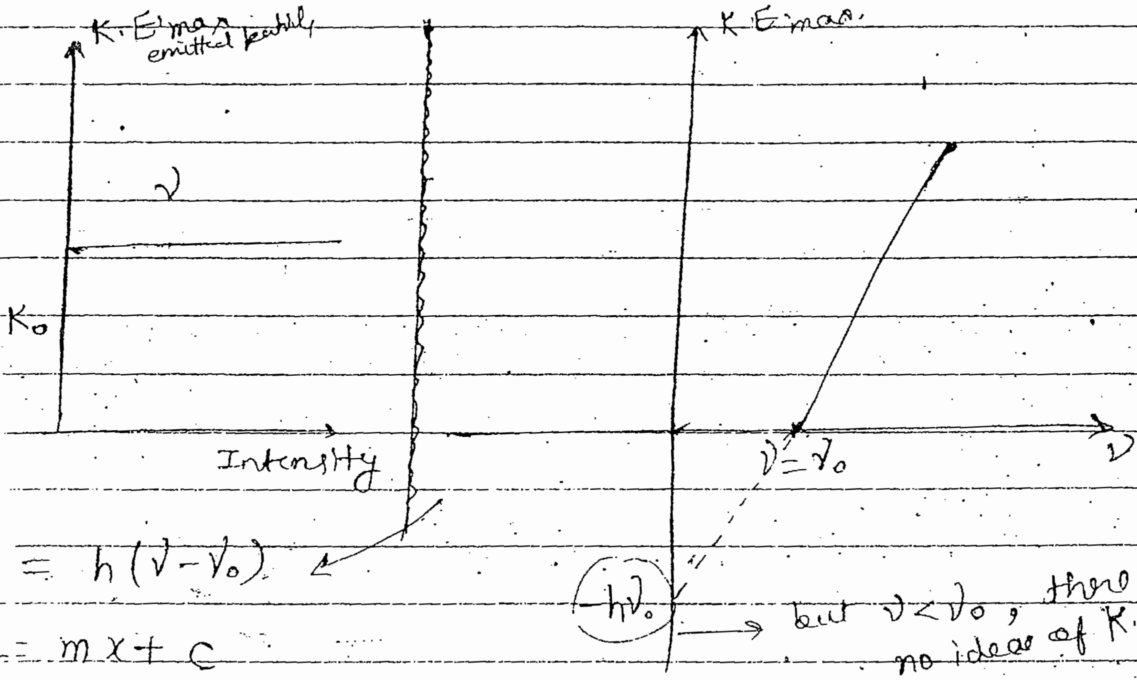
By inc. +ve pot^n, there is no change in e^-s produced.

$$V_0 = \frac{hc}{e}\left(\frac{1}{\lambda}\right) - \frac{W}{e}$$ $\boxed{V_0 \not\propto I}$

Date:

Page No.: 3

The pot^n at which photocurrent ~~will be~~ becomes zero is called stopping pot^n. & We can measure the current by passing through a resistence.



$\tan\theta = \frac{hc}{e}$ $K.E = h(\nu - \nu_0)$

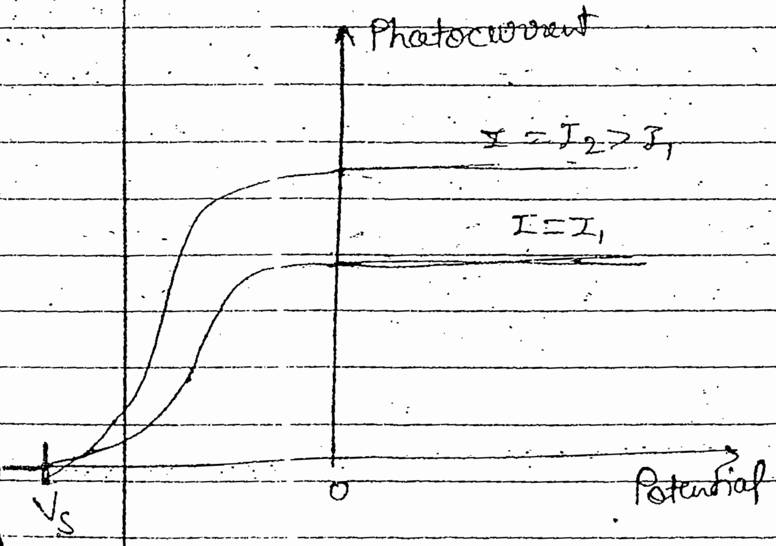
$y = mx + c$

$h\nu = mx, \; h\nu_0 = c$

$-h\nu_0 \rightarrow$ but $\nu < \nu_0$, there is no idea of K.E

$\nu > \nu_0 \rightarrow$ by solid line

$\nu < \nu_0 \rightarrow$ " dashed "



If ν is fixed the K.E. is also fixed.

If we apply +ve pot^n w.r.to +ve then photocurrent is constant

If apply −ve pot^n then e⁻ emitted are decreased so photocurrent decreases & at a particular pot^n it is zero called stoping Pot^n.

If we inc. the intensity & fix freq.

$I = I_2 > I_1$

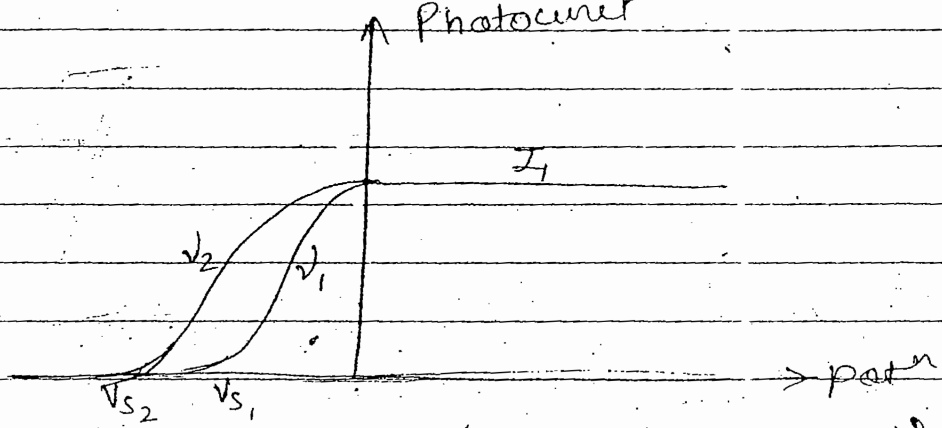
then no change in K.E.

but photocurrent inc.

stopping pot is related to the maxi

Date:
Page No.:

If we fix Intesity & chang the freq then there no change in photo current. Only change in stopping potn (related to K.E.)



(2) Compton Effect / Compton Scattering :- W.L. (wavelength) of scatter radiation is larger than the W.L. of incident radiation

(This can be explain by particle nature of x-ray photon) Exp. results shows that there is shift in W.L.

$$\lambda' = \lambda + \Delta\lambda$$

W.L. after scat. is larger than incident radia

Acc. to classical mech., if we incident free or x-ray on e^-, it will accelerate with the freq. of incident radiation.

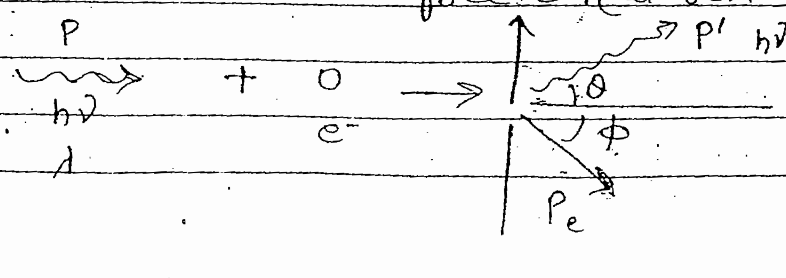
Accelerated charged particle emits radiation

Acc. to C.M., freq. of incident & emitted radiation are same. So No change in wave length.

This scat. is called Rayleigh Sca.

C.M. can't explain Comp. effect.

Acc. to Q.M., We include sca. of photon with free e^- (at rest but not always).



W.L. shift is depend on sca. angle not on intensity of incident radi freq. & W.L. of emitted photon is different so mom will be diff. $P = \frac{h}{\lambda}$

Elastic Sca. → Total Energy, K.E., & total line mom. of system are conserved.

Inelastic Sca. → Total E & T.L.M. is conserved but K.E. is not.

Apply Conservation to Total Energy :-

$$h\nu + m_0c^2 = h\nu' + K.E. + m_0c^2$$

$$= h\nu' + \sqrt{c^2p_e^2 + m_0^2c^4} - m_0c^2 + m_0c^2$$

$$h\nu + m_0c^2 = h\nu' + \sqrt{c^2p_e^2 + m_0^2c^4}$$

Conservation of K.E. :-

$$h\nu + 0 = h\nu' + \sqrt{c^2p_e^2 + m_0^2c^4} - m_0c^2$$

⇒ Conservation of T.E. & K.E. both are same.

Conservation of Linear Mom. :-

$$P = P' + P_e$$

$$\frac{h\nu}{c} = \frac{h\nu'}{c}\cos\theta + |P_e|\cos\phi$$

before sca. there is no vertical comp. of only Horizontal comp.

$$0 = \frac{h\nu'}{c}\sin\theta - |P_e|\sin\phi$$

Shift in W.L. $\boxed{\Delta\lambda = \frac{h}{m_0c}(1-\cos\theta)}$

$$\lambda' - \lambda = \frac{h}{m_0c}(1-\cos\theta)$$

$$\rightarrow \cancel{h}cm_0\left(\frac{\lambda'-\lambda}{\lambda\lambda'}\right) = \frac{h\cancel{=}}{\lambda\lambda'}(1-\cos\theta)$$

$$\boxed{\Delta\lambda = \frac{h}{m_0 c}(1-\cos\theta)}$$

Date :

Page No :

$$h\nu - h\nu' + m_e c^2 = \sqrt{c^2 p_e^2 + m_0^2 c^4}$$

$$\frac{hc}{\lambda} - \frac{hc}{\lambda'} + m_e c^2 = \text{"} \qquad \text{—— (A)}$$

We have to eleminate the mom. of e^- (P_e).

$$P = P' + P_e \Rightarrow P_e = P - P'$$

$$P_e^2 = (\bar{P} - P')\cdot(P - P')$$

$$= P^2 + P'^2 - 2P.P'$$

$$= \left(\frac{h}{\lambda}\right)^2 + \left(\frac{h}{\lambda'}\right)^2 - 2\frac{h}{\lambda}\frac{h}{\lambda'}\cos\theta$$

Put this in (A),

squaring(A),

$$\cancel{\left(\frac{hc}{\lambda}\right)^2} + \cancel{\left(\frac{hc}{\lambda'}\right)^2} + \cancel{m_0^2 c^4} = c^2 p_e^2 + \cancel{m_0^2 c^4}$$

$$- \frac{2hc}{\lambda}\frac{hc}{\lambda'} + \frac{2hc}{\lambda}m_0c^2 - \frac{2hc}{\lambda'}m_0c^2$$

$$\cancel{2c^2\left(\frac{1}{\lambda^2} - \frac{1}{\lambda'^2}\right) = 2hc} \qquad \text{"} \qquad = c^2\left[\cancel{\left(\frac{h}{\lambda}\right)^2} + \cancel{\left(\frac{h}{\lambda'}\right)^2} - 2\frac{h}{\lambda}\frac{h}{\lambda'}\cos\theta\right]$$

$$\Rightarrow \cancel{2hcm_0c^2\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = \frac{2c^2h^2}{\lambda\lambda'}(1-\cos\theta)}$$

$$\Rightarrow \Delta\lambda = \frac{h}{m_0 c}(1-\cos\theta) \quad , \theta \rightarrow \text{sca. angle}$$

shift * sca. ind on angle θ. — (W.L. shift before sca. depends on θ), ~~independent of~~

* Comp. W.L. shift depans on angle & λ intensity, freq

W.L. $(\theta, \nu, I, \lambda)$ i.e. after sca.

in sca. λ' depand on sca. angle, W.L. & freq. (θ, λ, ν) & indopendent on I.

$$\lambda_c = \frac{h}{m_0 c} = 2.43 \times 10^{-12} m$$

This is Compton W.L for e^-.

~~For $\theta = 0$~~, ~~i.e. e^- sca. in same dirn of incidence~~

then $\boxed{\Delta\lambda = 0}$

for $\theta = \pi/2$, $\boxed{\Delta\lambda = \lambda_c}$ for $\theta = \pi$ $\boxed{\Delta\lambda = 2\lambda_c}$

$\Delta\lambda = 2\lambda_c$ is the maxi. W.L. shift. Then correspon change in energy change is also maxi.

$E = \frac{hc}{\lambda} - \frac{hc}{\lambda'} = \frac{hc}{\lambda\lambda'}(\Delta\lambda)$ $\Delta\lambda \rightarrow$ maxi.
$E \Rightarrow$ maxi.

$\boxed{E = h(\nu - \nu')}$ $\left\{ \begin{array}{l} h\nu = h\nu' + E \\ E = h(\nu - \nu') \end{array} \right\}$

Comp. W.L. $\propto \frac{1}{m}$

As $m \uparrow$ then Comp. W.L. $\downarrow$.

e^- is considered free bcz $\rightarrow$

$\rightarrow$ ~~Applied~~ Supplied engy Engy of $e^- >$ B.E. then neglect B.E. so e^- can treat as fr

$\rightarrow$ If compare C.W.L. for e^- & atom then

$\frac{\lambda_{ce}}{\lambda_{cA}} = \frac{m_A}{m_e}$ $m_e \rightarrow$ neglesible than m_A

In case of Bind e^- the C.W.L shift is neglisible.

Problem :- In Compton sca. an incoming photon of wave length $\lambda_0 = \frac{3h}{2mc}$, ($h \rightarrow$ Plank const. $c \rightarrow$ speed of light in vaccum) is scattered at an angle 180° then the momentum corresponding to recoil e^- is

a. $\frac{20}{21} mc$ b. $\frac{8}{21} mc$ c. $\frac{21}{20} mc$ D. $\frac{16}{21} mc$

Sol^n :- Mom. conservation eq^n

$\boxed{p = p' + p_e}$

$\frac{h}{\lambda_0} = \frac{h}{\lambda'} + p_e \Rightarrow p_e = \frac{h}{\lambda_0} - \frac{h}{\lambda'}$

$p_e = \frac{2\cancel{h}mc}{3\cancel{h}} - \frac{2\cancel{h}mc}{7\cancel{h}}$

Date :
Page No. :

$$\lambda' = \lambda_0 + \Delta\lambda$$
$$= \frac{3h}{2mc} + \frac{h}{mc}(1-\cos\theta) = \frac{3h}{2mc} + \frac{2h}{mc}$$
$$= \frac{h}{mc}\left(\frac{3}{2}+2\right) = \frac{7h}{2mc}$$

$$P_e = mc\left(\frac{2}{3} - \frac{2}{7}\right) = \frac{8}{21}mc$$

for vector

$$P_e^2 = p^2 + p'^2 - 2pp'\cos\theta$$

$$p = \frac{h}{\lambda_0} = \frac{h\, 2mc}{3h} = \frac{2}{3}mc$$

$$p' = \frac{h}{\lambda'} = \frac{h\, 2mc}{7h} = \frac{2}{7}mc$$

$$P_e = P - P' \Rightarrow P_e^2 = (P-P')(P-P')$$
$$P_e^2 = P^2 + P'^2 - 2PP'\cos\pi$$
$$P_e^2 = P^2 + P'^2 + 2PP' = (P+P')^2$$
$$P_e = P + P'$$

$$P_e = \frac{2}{3}mc + \frac{2}{7}mc = mc\left(\frac{2}{3}+\frac{2}{7}\right) = \frac{20}{21}mc$$

(a) is correct.

* Old Q.T. → Radiation's particle nature.
New Q.T. → Particle's wave nature.

Problem :- In the scattering of x-rays with free e^- at rest the fractional ~~shift~~ loss in energy of a photon is

a. $\frac{\lambda+\Delta\lambda}{\Delta\lambda}$ b. $\frac{\lambda}{\lambda+\Delta\lambda}$ c. $\frac{\Delta\lambda}{\lambda}$ d. $\frac{\Delta\lambda}{\lambda+\Delta\lambda}$

(λ → incident w.L.)

In comp. sca., after sca. w.l. inc. or shift by $\Delta\lambda$, i.e. photon energy inc.

$$\cancel{\Delta\lambda = \frac{h}{mc}(1-\cos\theta)}$$

Sol^n:- Energy of a photon before sca. $E = \frac{hc}{\lambda}$

after sca. $E' = \frac{hc}{\lambda'}$

loss in engy

$$E - E' = \frac{hc}{\lambda} - \frac{hc}{\lambda'} \quad \& \quad \lambda' = \lambda + \Delta\lambda$$

$$= hc\left(\frac{1}{\lambda} - \frac{1}{\lambda+\Delta\lambda}\right)$$

$$E - E' = hc\left(\frac{\lambda+\Delta\lambda-\lambda}{\lambda(\lambda+\Delta\lambda)}\right) = \frac{hc\,\Delta\lambda}{\lambda(\lambda+\Delta\lambda)}$$

fractional loss in energy $= \frac{E-E'}{E} = \dfrac{\frac{hc}{\lambda}\,\frac{\Delta\lambda}{\lambda(\lambda+\Delta\lambda)}}{\frac{hc}{\lambda}}$

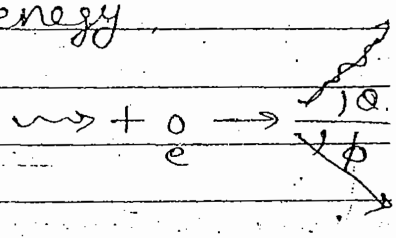
$$\boxed{\frac{E-E'}{E} = \frac{\Delta\lambda}{\lambda+\Delta\lambda}}$$

∴ (d) is correct.

Note:- K.E. given to the e^- = ?

then K.E. = change in enegy.

$$E + m_oC^2 = E' + K.E. + m_oC^2$$



$$\boxed{K.E. = E - E'}$$

Prob:- Consider a photon that scatteres from an e^- at rest if the compton W.L. shift is observed to be triple the W.L. of the incident photons & if the photon scatters at 60° then calculate

(i) The W.L. of the incident photon

Date:
Page No.:

(ii) The energy given to the recoiling e^-.

$\Delta\lambda = 3\lambda$

$\theta = 60°$

(i) $\Delta\lambda = \frac{h}{m_0 c}(1-\cos\theta)$

$3\lambda = \frac{h}{m_0 c}(1-\cos 60) = \frac{h}{m_0 c}(1-\frac{1}{2})$

$3\lambda = \frac{1}{2}\frac{h}{m_0 c} \Rightarrow \lambda = \frac{1}{6}\frac{6.626 \times 10^{-34}}{9.1 \times 10^{-31} \times 3 \times 10^{8}}$

$\lambda = \frac{6.626}{163.8} \times 10^{-11}$

$\lambda = 0.04029 \times 10^{-11} = 4.029 \times 10^{-13} m$

(ii) $E = \frac{hc}{\lambda} - \frac{hc}{\lambda'}$

$\lambda' = \lambda + \Delta\lambda \Rightarrow \lambda' = \lambda + 3\lambda \Rightarrow \lambda' = 4\lambda$

$\lambda' = 4 \times \frac{h}{6 m_0 c} = \frac{2h}{3 m_0 c}$

$E = E' + K.E.$

$K.E. = E - E'$

$= \frac{hc}{\lambda} - \frac{hc}{\lambda'} = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = \frac{hc}{\lambda\lambda'}\Delta\lambda$

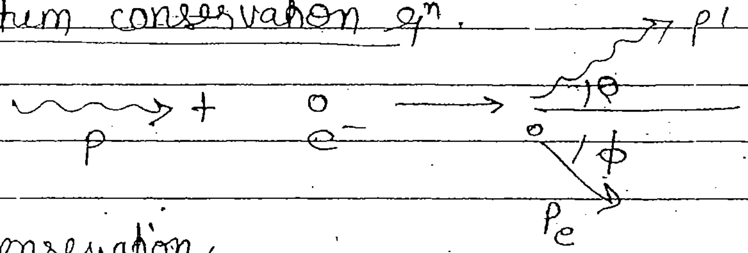
$= \frac{6.626 \times 10^{-34} \times 3 \times 10^{8} \times 3 \times 4.029 \times \cancel{10^{-13}}}{4.029 \times 10^{-13} \times 4 \times 4.029 \times \cancel{10^{-13}}}$

$= \frac{240.26539}{64.93136} \times 10^{-13}$ $-34+8+13$

$= 3.7003 \times 10^{-13} J = \frac{3.7003 \times 10^{-13}}{1.6 \times 10^{-19}} = 2.312 \times 10^{6} eV$

$= 2.3$ MeV

Note :- To calculate the angle ϕ, use the concept of momentum conservation eqn.



Mom. Conservation,

$$P = P' + P_e$$

$$\frac{h}{\lambda} = \frac{h}{\lambda'}\cos\theta + |P_e|\cos\phi$$

In vertical component e. & photon are in opposite dirn (so minus)

$$0 = \frac{h}{\lambda'}\sin\theta - |P_e|\sin\phi$$

Using these two eqn, calculate ϕ,

$$|P_e|\sin\phi = \frac{h}{\lambda'}\sin\theta$$

$$|P_e|\cos\phi = \frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta$$

dividing,
$$\tan\phi = \frac{\frac{h}{\lambda'}\sin\theta}{\frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta}$$

$$\boxed{\tan\phi = \frac{\sin\theta}{\frac{\lambda'}{\lambda} - \cos\theta} = \frac{\sin\theta}{\frac{|P|}{|P'|} - \cos\theta}}$$

$\phi \to$ angle at which e⁻ recoil after sca.

Problem :- Consider an X-ray beam with $\lambda = 1\,\text{Å}$ and a γ-ray beam with $\lambda = 1.88 \times 10^{-2}\,\text{Å}$. If the radiation scattered from e⁻ in Cs^{137} sample at $90°$ to the incident dirn.

(i) What is the compton W.L. shift in each case (for X-ray & γ-ray)

(ii) $X \to 0.245$ KeV
$\gamma \to 378$ KeV

(iii) Date: 2.4 %, 57
Page No.:

(ii) What K.E. is given to Recoiling e^- in each case
(iii) What % of the incident photon energy is lost in each case

for X-ray $\lambda = 1\,\mathring{A} = 10^{-10}$ m
~~γ-ray $\lambda = 1.88 \times 10^{-2}\,\mathring{A}$~~
$\theta = 90^\circ$

(i) $\lambda' - \lambda = \frac{h}{m_0 c}(1-\cos\theta)$

for x-ray $\lambda' = \lambda + \frac{h}{m_0 c}(1-\cos\theta)$

$= 10^{-10} + \frac{h}{m_0 c}(1-\cos 90^\circ)$

$= 10^{-10} + \frac{6.626 \times 10^{-34}}{9.1 \times 10^{-31} \times 3 \times 10^{8}}(1-0)$

$= 10^{-10} + 0.2417 \times 10^{-11}$

$= 10^{-10}\left(1 + \frac{0.2417}{10}\right) = 10^{-10}(1+0.02417)$

~~$\lambda' = 0.97583 \times 10^{-10} = 0.97583\,\mathring{A}$~~

$\Delta\lambda = \lambda' - \lambda = $ ~~10^{-10}~~ $\qquad \lambda' = 1.02417 \times 10^{-10}$

$\Delta\lambda = $ ~~$1 - 0.97583\,\mathring{A}$~~ $(1.0247 - 1) \times 10^{-10}$

$\Delta\lambda = $ ~~$0.02417\,\mathring{A}$~~

$\Delta\lambda = 0.02417\,\mathring{A}$ Ans

for γ-ray, $\Delta\lambda = 0.02417\,\mathring{A}$ becz W.L. shift independent of source.

(ii) $K.E. = E - E' = \frac{hc}{\lambda} - \frac{hc}{\lambda'} = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)$

~~for x-ray~~, $K.E. = hc($

Date :
Page No. :

8

$$K.E. = 6.67 \times 10^{-34} \times 3 \times 10^{8} \left[\frac{1}{10^{-10}} - \frac{1}{1.02417 \times 10^{-10}} \right]$$

$$= 20.01 \times 10^{-16} \left[\frac{1.02417 - 1}{1.02417} \right] = \frac{20.01 \times 0.02417 \times 1}{1.02417}$$

$$= \frac{0.47223 \times 10^{-16}}{1.6 \times 10^{-19}} e = 0.295 \times 10^{3} = 0.295 \text{ keV}$$

for γ-ray, $\lambda' = \lambda + \Delta\lambda$

$$\lambda' = \cancel{1.88 \times 10^{-12}} \quad 0.0188 + 0.02417$$

$$\lambda' = 0.04297 \text{ Å}$$

$$K.E. = E - E' = \frac{hc}{\lambda} - \frac{hc}{\lambda'} = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)$$

$$= 20.01 \times 10^{-16} \left[\frac{1}{0.0188} - \frac{1}{0.04297} \right]$$

$$= 20.01 \times 10^{-16} (53.19149 - 23.27205)$$

$$= 20.01 \times 29.92 \times 10^{-16} = 598.6992 \times 10^{-16} \text{ J}$$

$$= \frac{598.6992 \times 10^{-16}}{1.6 \times 10^{-19}} = 374.187 \times 10^{3} e$$

$$= 374.187 \text{ keV}$$

(iii) % of incident photon energy loss

for x-ray, $\frac{E - E'}{E} = ?$

$$E = \frac{hc}{\lambda} = 20.01 \times 10^{-16} = 12.50625 \text{ keV}$$

$$= 0.02359$$

$$\% \frac{E - E'}{E} \times 100 = 2.359 \simeq 2.4\ \%$$

for γ-ray, $E = \frac{hc}{\lambda} = \frac{20.01 \times 10^{-16}}{0.0188}$

$$= 1064.3617 \times 10^{-16}$$

$$\% \frac{E - E'}{E} = \frac{598.6992}{1064.3617} \times 100 = 0.5625$$

$$= 56.25\ \%$$

Que:- In the photoelectric effect experiment if a radiation of W.L. λ is incident on a material then stopping potential V_0 is observed. If another radiation of W.L. twice λ (2λ) is incident on the same material then stopping potⁿ change to

a) $\frac{V_0}{2}$ b) $2V_0$ c) $V_0 - \frac{hc}{e\lambda}$ d) $V_0 + \frac{hc}{2e\lambda}$

e.) $V_0 - \frac{hc}{2e\lambda}$ f.) $V_0 + \frac{hc}{2e\lambda}$

$\lambda \longrightarrow V_0$ & $2\lambda \longrightarrow$

for λ, $V_0 = \frac{hc}{e}\left(\frac{1}{\lambda}\right) - \frac{W}{e}$

2λ, $V_0 = \frac{hc}{e}\frac{1}{2\lambda} - \frac{W}{e}$

$E = K.E. + W$

$\frac{hc}{\lambda} = K.E._1 + W$ —— ①

for same material, work fuⁿ W is same

for another radiation, $\frac{hc}{2\lambda} = K.E_2 + W$ —— ②

$K.E. = qV_0$

① $\Rightarrow$ $\frac{hc}{\lambda} = eV_0 + \cancel{W}$

② $\Rightarrow$ $\frac{hc}{2\lambda} = eV_s + \cancel{W}$

$$eV_0 - eV_s = \frac{hc}{\lambda} - \frac{hc}{2\lambda}$$

$$eV_s = eV_0 - \frac{hc}{\lambda} + \frac{hc}{2\lambda}$$

$$\boxed{V_s = V_0 - \frac{hc}{2e\lambda}}$$

(e) is correct.

Prob :- When light of a given W.L. is incident on a matellic surface, the stopping potⁿ for the photoelectron is 3.2 V. If a IInd light source whose W.L. is double that of the Ist source is used the stopping potⁿ drops to 0.8 V. Calculate

(i) the W.L. of the Ist radiation

(ii) the work funcⁿ & cutt off freq. of the metal

$$V_0 = 3.2\ V \quad , V_S = 0.8\ V$$

$$\lambda_{II} = 2\lambda_I$$

$$V_0 - V_S = V_0 - 0.8\ V = 3.2 - 0.8\ V = 2.4\ V$$

For I source, $\dfrac{hc}{\lambda} = eV_0 + W$ $\qquad \dfrac{hc}{\lambda} = e \times 3.2 + W$

II , $\dfrac{hc}{2\lambda} = eV_S + W$ $\qquad \dfrac{hc}{2\lambda} = e \times 0.8 + W$

$\lambda = ?$

$$hc\left(\frac{1}{\lambda} - \frac{1}{2\lambda}\right) = e(V_0 - V_S)$$

$$\frac{hc}{2\lambda} = e\left[3.2 - 0.8\right]$$

$$\frac{hc}{2\lambda} = e\,(2.4)$$

$$\lambda = \frac{hc}{2 \times e \times 2.4} = \frac{6.626 \times 10^{-34} \times 3 \times 10^{8}}{2 \times 1.6 \times 10^{-19} \times 2.4}$$

$$\lambda = 2.6 \times 10^{-34+27}$$

$$\lambda = 2.6 \times 10^{-7}\ m$$

(ii) $\dfrac{hc}{\lambda} = eV_0 + W$

$= e \times 3.2 + W$ $\qquad W = 2.56 \times 10^{-7}\ m$

$W = h\nu_0 \Rightarrow \nu_0 = \dfrac{W}{h}$

Date :
Page No. :

* In photoelectric effect → there is no photon after sca. but in compton effect → there is photon after sca.
i.e. In P.E.E → photon incident on material absorb completely
~~In Compton → some energy of photon is absorb~~

If in compton, e^- absorb the photon completely then no photon after sca. only e^- remain.

$$\overset{h\nu}{\rightsquigarrow} + \underset{e^-}{\circ} \longrightarrow \underset{P_e}{\circ\, e}$$

$$\frac{h}{\lambda} = \frac{h\nu}{c}$$

Then conservation of energy & mom. apply,

- $h\nu + m_oc^2 \longrightarrow K.E. + m_oc^2 = \sqrt{P_e^2c^2 + m_o^2c^4} - m_oc^2 + m_oc^2$

$$h\nu + m_oc^2 = \sqrt{P_e^2c^2 + m_o^2c^4} \quad \text{--- (1)}$$

- $P_{ph} = P_e$

$$\frac{h\nu}{c} = |P_e| \quad \text{--- (2)}$$

Conservation of ~~po~~ mom. is not conserved

$$(1) \Rightarrow \frac{h\nu}{c} + \frac{m_oc^2}{c} = \frac{1}{c}\sqrt{P_e^2c^2 + m_o^2c^4}$$

$$\frac{h\nu}{c} \neq |P_e|$$

i.e. If free e^- at rest do not absorb photon completely due to the conservation of linear mom.
So In P.E.E e^- should be bind.

Note :- In Compton → e^- should be free (photon interact with e^-)
Photoelectric → " " bound, (with atom)
[In Pair P. with nucleus]

Pair Production :-

Pair → particle & its antiparticle pair generate with the help of photon.

The theory of relativistic Q.M. predicts the existance of positron. This is the antiparticle of e^- having same mass but equal & opposite charge.

*Conversion of Energy into mass (or matterisation of radiation) :-

In Pair P., radiation interact with nucleus. Int happen b/w those which have comparable size.

$$h\nu \rightsquigarrow + \underset{N}{O} \longrightarrow \underset{N}{O} + \begin{matrix} e^- \nearrow \\ e^+ \searrow \end{matrix}$$

Radiation intract always with matter.

Any particle & antiparticle can produce correspondi to the enrgy of inci. radiation. but mostly we consider $e^- - e^+$ pair bcoz e^- is lightest particl

Minimum energy req. to generate pair is $\boxed{1.02 \text{ MeV}}$

Energy Conservation,

$$h\nu + m_N c^2 = K_N + m_N c^2 + K_{e^-} + m_e c^2 + K_{e^+} + m_e c^2$$

$$h\nu = K_N + K_{e^-} + K_{e^+} + 2 m_e c^2$$

$$\boxed{h\nu = K_N + 2\gamma m_0 c^2}$$

$E = mc^2$

$m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$

$= \gamma m_0$

$\Rightarrow E = \gamma m_0 c^2$

(particle & antiparticle have almost same energy. The energy diff b/w them is neglesible)

* Pair production is not possible in vaccoume or empty or free space.

{ e^- has -ve charge & e^+ has +ve, Nucleus is +ve charge. by interacting e^- & nucleus . e^- energy ...

Date :
Page No. :

$h\nu \rightsquigarrow \quad \longrightarrow \quad e^-, e^+$ (at angle θ)

In vacuume, energy con. $h\nu = 2\gamma m_0 c^2$ — (i) sca. angle considered sa mom. cons.

mom con. $\frac{h\nu}{c} = 2|P_e|\cos\theta$ — (ii)

$= 2\gamma m_0 v \cos\theta$ $P =$ $= \gamma$

$\cos\theta = +1$ to -1

Maxi value of $\cos\theta = +1$ i.e. $\theta = 0°$.

for $\theta = 0$, $\frac{h\nu}{c} = 2\gamma m_0 v$ (maximum value)

So $\boxed{\frac{h\nu}{c} \leq 2\gamma m_0 v}$ by mom. conservation

(i) $\Rightarrow$ $\frac{h\nu}{c} = \frac{2\gamma m_0 c^2}{c}$

$\frac{h\nu}{c} = 2\gamma_0 m_0 c \times \frac{v}{v} = 2\gamma m_0 v \frac{c}{v}$

If $v = c$ then

$\frac{h\nu}{c} = 2\gamma m_0 v$

but for any material particle $v < c$ so $\left\{ \frac{c}{v} > \right.$

$\boxed{\frac{h\nu}{c} > 2\gamma m_0 v}$ by energy con.

There are two contradictory results, so mom. & energy are not conserved simultaneously. Pair production is not possible in vaccum due to non conservation in linear mom. & energy simultaneously.

<u>Role of Nucleus</u> → to absorb the extra mom. & conserve the linear mom. & energy simultaneously.

Pair Anihilation :- This is the inverse of Pair production in which particle & antiparticle anihilates. Conversion of mass into energy.

After annihilating particle & antiparticle give some energy.

e^- & e^+ move in same dir^n for anihilation

(1) $e^- \searrow$, $e^- \nearrow$ $\longrightarrow$ $\rightsquigarrow$

(2) $e^- \rightarrow \leftarrow e^+ \longrightarrow \rightsquigarrow$

In (2), mom. of $e^- \rightarrow P_e$ & mom. of e^+ ~~is~~ P_e ~~&~~ move with ~~equal & opposite mom~~

So $P_e - P_e = 0$

So mom. of photon should also be 0.

In case of single photon, mom. of photon is $\neq 0$.

So for pair anihilation, two photons are req.

So $e^- \rightarrow \leftarrow e^+ \longrightarrow \overset{P_1}{\rightsquigarrow} \quad \overset{P_2}{\leftsquigarrow}$

There may be $P_1 = P_2 = 0$

or $P_1 = -P_2$

i.e. In Pair anihilation at least 2 photon produce.

Problem :- How many positrons can ~~take~~ a 200 MeV photon produce.

Pair $\rightarrow e^- - e^+$

Min. energy to generate a singe pair $= 1.02$ MeV

$$\text{No. of positrons} = \frac{200 \text{ MeV}}{1.02 \text{ MeV}}$$

$$= 195 \text{ Positrons}$$

Date :
Page No. :

Prob 1- An e^- & e^+ at rest producing three photons afte anihilating. Find the energy of III-photon if energy of 2 of the photons are 0.2 MeV & 0.3

Addition of rest mass enrgy of e^- & e^+ = 1.02

$1.02 = E_1 + 0.2 + 0.3$

$E_1 = 1.02 - 0.5 = 0.52$ MeV.

Wave Nature of Particle :-

De-Broglie Concepts :- Acc. to this concept any material particle (neutral or charge) in motion is equalent to wave.

The wavelength is given by

$$\lambda = \frac{h}{|p|}$$

And corresponding wave vector

$$k = \frac{2\pi}{\lambda}\hat{n}$$

$\hat{n}$ → unit vector along the dir^n of motion. if particle move in x-dir^n then wave will propogate in x-dir^n.

These relations are called De Broglie Concepts. This concepts experimantely ~~explained~~ varified by Devision & Germer. And Thomson exp.

for any Quantum mechanical particle, we can calculat the De Broglie relations.

Compare the De broglie wavelength with's size of the system. to cheak the wave nature.

~~If~~ DeB.WL. is comparable (or of the order of) the size

$\lambda \sim$ size of the system

of the system then wave nature can be detected & system can behave like a wave.
If D.B WL is very small compare to size the we can not detect the wave nature. & system can not behave like a wave.

Que:- Calculate the De Broglie Wavelength for
(i) a proton of Kinetic energy 70 MeV.
(ii) a 10 gm ball moving at 900 m/sec (velocity).

(i) $\lambda = \frac{h}{|p|}$

$p = \sqrt{2mE}$, $E = 70$ MeV

$= \cancel{\sqrt{2 \times 938 \times 70}}$

$= \sqrt{2 \times 1.67 \times 10^{-19} \times 70 \times 10^{6} \times 1.6 \times 10^{-19}}$

$\lambda = \frac{6.626 \times 10^{-34}}{\quad}$

$\lambda \sim 10^{-15}$ m

(ii) $\lambda = \frac{h}{mv}$, $v = 900$ m/sec , $m = 10$ gm

$\lambda =$

$\lambda \sim 10^{-34}$ m

Size of nucleus is 10^{-15} m. In (i) case $\lambda \sim 10^{-15}$ m It is comparable to size but in (ii) case $\lambda \sim 10^{-34}$ is very-2 small as compare to its size. (negligible) ... we can't detect the wave nature at

Date :
Page No. :

* Hence we can calculate the De-broglie W.L. for microscopic systems. Not for Microscopic.

Q.2:- Calculate the De Broglie W.L. for an e^- accelerated to the pot^n difference V.

K.E. = charge × pot^n

$$\boxed{K.E. = eV}$$

for Non-relativistic $(v<c)$, $K.E. = \frac{p^2}{2m}$

for relativistic $(v\simeq c)$, $E = K.E. + m_0c^2$
$= \sqrt{c^2p^2 + m_0^2c^4}$

De-Broglie W.L. may be relativistic or non-rel^v.

Non-Relativistic De Broglie W.L. ;

$$\lambda_{Non} = \frac{h}{|p|} = \frac{h}{\sqrt{2mE_k}}$$ $E_k \to K.E.$

$$\lambda_{Non} = \frac{h}{\sqrt{2meV}} = \frac{12.28 \times 10^{-10}}{\sqrt{V}} m$$

$$= \frac{12.28}{\sqrt{V}} \text{Å}$$

$$\boxed{\lambda_{Non} = \sqrt{\frac{150}{V}} \text{Å}}$$ This is only for e^-

$$\boxed{\lambda_{Non\text{-}rel} = \frac{h}{\sqrt{2mE_k}}}$$ is valid for all the particles,

Relativistic De-B. W.L.

$$\lambda_{rel} = \frac{h}{|p|}$$

k.E. + Rest Energy

$$E = K.E. + m_0c^2 = \sqrt{c^2p^2 + m_0^2c^4}$$

$$c^2p^2 + m_0^2c^4 = (eV + m_0c^2)^2$$

$$c^2p^2 + \cancel{m_0^2c^4} = (eV)^2 + \cancel{m_0^2c^4} + 2eVm_0c^2$$

$$c^2p^2 = 2eVm_0c^2\left[1 + \frac{eV}{2m_0c^2}\right]$$

$$p^2 = 2eVm_0\left[1 + \frac{eV}{2m_0c^2}\right]$$

$$\therefore \boxed{\lambda_{rel} = \frac{h}{\sqrt{2eVm_0\left[1 + \frac{eV}{2m_0c^2}\right]}}}$$

Both De.B. W.L. are different for Relativisic & Non-rel^v.

Compare Relativistic & Non-Rel^v results :-

$$\lambda_{rel} = \frac{h}{\sqrt{2eVm_0\left(1 + \frac{eV}{2m_0c^2}\right)}}$$

$$\lambda_{Non-rel} = \frac{h}{\sqrt{2meV}}$$

If we approximate $\frac{eV}{2m_0c^2} << 1$

i.e. $eV << 2m_0c^2$

If K.E. term is very-2 less than Rest mass energy then we use Non-rel^v D.B. W.L.

If both are comparable then we use relativistic

$$E = \sqrt{c^2p^2 + m_0^2c^4}$$

Also we have $\boxed{E = cp}$ (Extreme rel^v)

We have 3 cases - Rel^v & Non Rel^v & Extreme Rel^v (for photon - Extreme Rel^v)

If Rest mass energy term is large - Non-Relativity

" " " " small - Relativity

$cp << m_0c^2 \rightarrow$ Non & $cp >> m_0c^2 \rightarrow$ Rel^v

Date :
Page No. :

Q.3 :- Calculate the De broglie W.L. for a material particle of mass m that is in thermal equili at temp T.

$$\lambda = \frac{h}{|p|}$$ This is valid for photon as well as for material particle.

$$\lambda = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{2mKT}} \Rightarrow \lambda = \frac{h}{\sqrt{2mKT}}$$

(Most probable D.B. WL

In 3 dim motion, there are 3 types of speed mean, root mean & root mean square in thermodynamics, In 3-Dim, In M.W. Boltzman statistics $KE = \frac{1}{2}k$

for 1 degree of freedom $K.E. = \frac{1}{2}KT$

$$\therefore \lambda = \frac{h}{\sqrt{2m \times \frac{1}{2}KT}} = \frac{h}{\sqrt{mKT}}$$

for 2 D.O.F. $K.E. = KT$

$$\lambda = \frac{h}{\sqrt{2mKT}}$$

for 3 D.O.F. $K.E. = \frac{3}{2}KT$

$$\lambda = \frac{h}{\sqrt{2m \times \frac{3}{2}KT}} = \frac{h}{\sqrt{3mKT}}$$

Most probable $K.E. = \frac{1}{2}mv^2$

$$v_p = \sqrt{\frac{2KT}{m}} \quad (K.E. = KT)$$

$$v_{rms} = \sqrt{\frac{3KT}{m}} \quad (K.E. = \frac{3}{2}KT)$$

Most Probable DB. WL. $\lambda = \frac{h}{\sqrt{2mKT}}$ & Avg. DB. WL. $\lambda = \frac{h}{\sqrt{3mKT}}$

Q.4 :- The speed of an e^- whose De-Broglie W.L. is equal to its compton W.L.

(a) c (b) $\frac{c}{\sqrt{2}}$ (c) $\frac{c}{2}$ (d) $\frac{c}{3}$

$c \rightarrow$ Speed of light

De Broglie W.L. $\lambda = \frac{h}{|p|}$

Compton W.L. $\lambda = \frac{h}{m_0 c}$

$$\frac{h}{p} = \frac{h}{m_0 c} \Rightarrow \frac{h}{mv} = \frac{h}{m_0 c}$$

$\frac{h}{p} = \frac{h}{mv}$

$$\frac{h}{\frac{m_0 v}{\sqrt{1-v^2/c^2}}} = \frac{h}{m_0 c}$$

In $p = mv$ can't take the ~~relativistic~~ rest mass, We take the relv mass.

$$\frac{\sqrt{1-v^2/c^2}}{v} = \frac{1}{c}$$

$$\Rightarrow 1 - \frac{v^2}{c^2} = \frac{v^2}{c^2} \Rightarrow 1 = \frac{2v^2}{c^2} \Rightarrow \frac{v}{c} = \sqrt{\frac{1}{2}}$$

$$\Rightarrow \boxed{v = \frac{c}{\sqrt{2}}}$$

Q.5 :- Show that the Debroglie W.L. associated with a particle of rest mass m_0 & ~~K.E.~~ K is given by

$$\lambda = \frac{ch}{\sqrt{K(K+2m_0c^2)}}$$

$$\sqrt{c^2p^2 + m_0^2c^4} = K + m_0c^2$$

$$c^2p^2 + m_0^2c^4 = (K + m_0c^2)^2$$

$$c^2p^2 + \cancel{m_0^2c^4} = K^2 + \cancel{m_0^2c^4} + 2Km_0c^2$$

Date :
Page No. :

$$c^2p^2 = K^2 + 2m_o Kc^2$$

$$cp = \sqrt{K^2 + 2Km_oc^2}$$

$$\lambda = \frac{h}{p} = \frac{ch}{cp} = \frac{ch}{\sqrt{K^2+2Km_oc^2}}$$

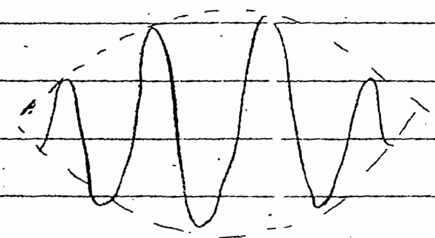
$$\boxed{\lambda = \frac{ch}{\sqrt{K(K+2m_oc^2)}}}$$ Proved

If Quantum particle is in motion, then velocity of wave & particle will be different. In that case De Broglie ~~tat~~ concept is not satisfied, i.e. We have to modify the De-Broglie concept,

Schrodinger modified the De broglie wave, If Q.P. is in motion, particle is not like a wave, it is like a group of wave.

Wave Packet :- a surperposition of a group of wave of slightely different W.L.s and velocities Amplitude & phases so choosen that they interfere constructivlely in small region of space i.e. in the neighbourhood of particle. & interefere ~~respectively~~ distructively elsewhere.

for Constructive Inerference, $y = y_1 + y_2$
distructive " $y = y_1 - y_2$



Date:
Page No.:

Wave packet is also called localised wave func^n or bound state.

There are 2 types of wave func^{n}s - localised & Non-locali

Note:- Non-localised wave func^n or scattering state or unbound state.

If wave func^n exist up to ∞ → Non-localised.

" " " " finite → Localised.

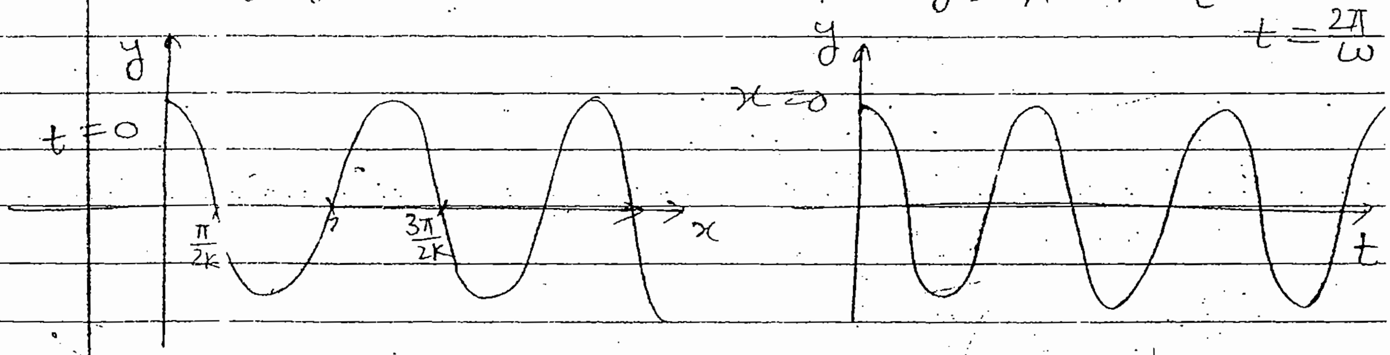
Localised wave func^n at ∞ = 0

Consider 2 waves

$$y_1 = A\cos(Kx - \omega t)$$

$$y_2 = A\cos\{(K+\Delta K)x - (\omega+\Delta\omega)t\}$$

Plot of $y = A\cos Kx$ $\qquad\qquad$ $y = A\cos\omega t$



The ~~kt~~ variation of plot will same but we get $T = \frac{2\pi}{\omega}$ the maxima.

for $t=0$, $\quad Kx = \frac{n\pi}{2}$ $\qquad n = 1, 3, 5, \ldots$

$$x = \frac{n\pi}{2K} \quad \text{(distance b/w 2 zeros)}$$

Matematically, $\quad Kx - \omega t = \text{const.}$

$$\boxed{\frac{dx}{dt} = \frac{\omega}{K} = v_p} \qquad v_p \to \text{phase vel.}$$

Date :
Page No. :

Distance travelled in t-time, $= \frac{\omega}{K} t$

Phase velocity is the velocity of individual wave & group " " " Wave packet.

Now, $y_1 = A\cos(Kx - \omega t)$

$y_2 = A\cos\{(K+\Delta K)x - (\omega+\Delta\omega)t\}$

By Constructive Interference,

$y = y_1 + y_2$

$$y = A\left[\cos\{(K+\Delta K)x - (\omega+\Delta\omega)t\} + \cos(Kx - \omega t)\right]$$

$$y = A \times 2\cos\left\{\left(\frac{K+\Delta K+K}{2}\right)x - \frac{(\omega+\Delta\omega+\omega)t}{2}\right\} \times \cos\left(\frac{\Delta K x}{2} - \frac{\Delta\omega t}{2}\right)$$

$$y = 2A\cos\left(\frac{\Delta K x - \Delta\omega t}{2}\right)\underbrace{\cos(K_0 x - \omega_0 t)}$$

~~Plot of this part will be~~ same as $y = A\cos(Kx - \omega t)$

$\Delta K x = \frac{n\pi}{2}$

$x = \frac{\pi}{2\Delta K}, \frac{3\pi}{2\Delta K}, \frac{5\pi}{2\Delta K} \dots$

$K_0 x = \frac{n\pi}{2}$

$x = \frac{\pi}{2K_0}, \frac{3\pi}{2K_0}, \frac{5\pi}{2K_0} \dots$

This K_0 is larger than ΔK (bcoz it is the difference)

$\frac{3\pi}{2K_0} - \frac{\pi}{2K_0} = \frac{\pi}{K_0}$ $K_0 > \Delta K$

$\frac{3\pi}{2\Delta K} - \frac{\pi}{2\Delta K} = \frac{\pi}{\Delta K}$

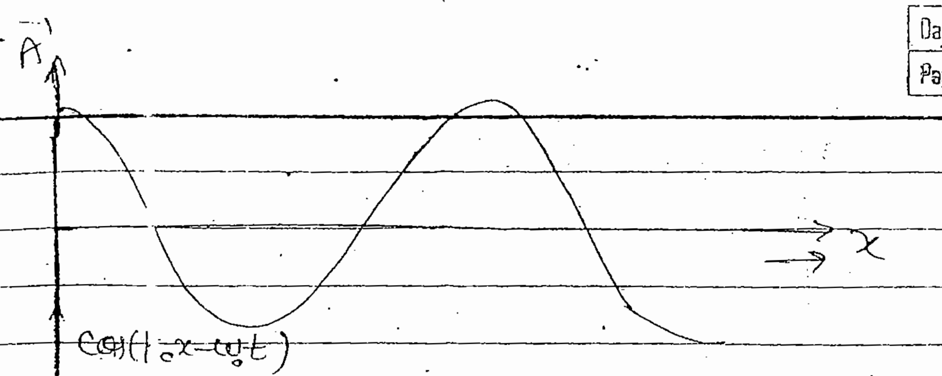
So W.L. for larger K_0 will be small &

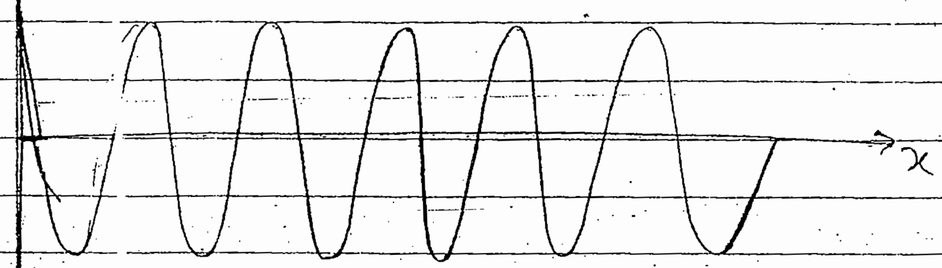
" " small ΔK " " large.

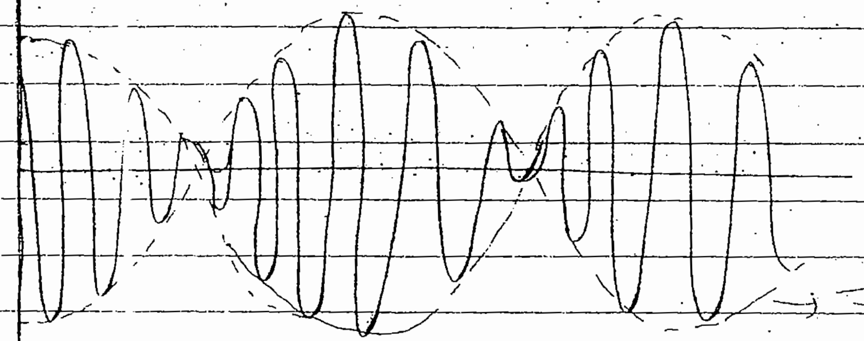
$\Delta K \rightarrow$ difference of 2 wave vectors.

$K_0 \rightarrow$ Sum of " " "

$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$







This spreading depend on medium

The velocity at which the wave packet propogate is Group velocity.

As time pases, the wave packet will further ~~in~~ depands upon velocity.

Group velocity is specified by $\frac{\Delta k x - \Delta \omega t}{2}$

$$\boxed{v_g = \frac{\Delta\omega}{\Delta k}} \quad \text{or} \quad \frac{d\omega}{dk}$$

The dir^n of phase velocity & group velocity is same in the dir^n of particle motion.

~~No~~ for Non-dispersive medium, $V_p = V_g$
so No displacement in wave packet bcz there is: No distortion in wave packet
for Dispersive medium, $V_p \neq V_g$ so there is displacement bcoz of spreading of wave packet.

Date :
Page No. :

For a signle particle $v_p \neq v$, it is $v_p > c$
Here special theory of relativity violets.
So single wave can never propogate. Only a group of wave (wave palcet) can propogate.
In Q.M. $\rightarrow$ No meaning of phase velocity.
~~So we interprete particle velocity is group~~ velocity.

$\Rightarrow$ To calculate the velocity of single particle we ~~take~~ calculate phase velocity but for interpritation, we interprete with group vel.

$$v_p = \frac{\omega}{k} \quad \& \quad v_g = \frac{d\omega}{dk}$$

We can change the v_p & v_g into energy,

$$v_p = \frac{\omega}{k} = \frac{\hbar\omega}{\hbar k} = \frac{E}{p}$$

$$v_g = \frac{d\omega}{dk} = \frac{dE}{dp}$$

Relation b/w v_p & v_g :-

$$v_g = \frac{d\omega}{dk}$$

$$= \frac{d}{dk}(k v_p) \quad \text{as } \omega = k v_p$$

$$v_g = v_p + k\frac{dv_p}{dk}$$

$$v_g = v_p + \hbar k \frac{dv_p}{dk\hbar}$$

$$v_g = v_p + p\frac{dv_p}{dp}$$

$k = \frac{2\pi}{\lambda}$ $p = \frac{h}{\lambda}$ $dp = -\frac{h}{\lambda^2}d\lambda$ $\lambda = \frac{h}{p}$ $\frac{2\pi}{k} = \frac{h}{p}$ $\frac{k}{2\pi} = \frac{p}{h}$ $p = \frac{hk}{2\pi} = \hbar k$

$$k = \frac{2\pi}{\lambda} \Rightarrow dk = \frac{-2\pi}{\lambda^2} d\lambda$$

$$\boxed{v_g = v_p - \lambda \frac{dv_p}{d\lambda}}$$

v_g & v_p depends on $\frac{dv_p}{d\lambda}$, $\frac{dv_p}{dP}$, $\frac{dv_p}{dk}$.
group vel., v_g may be smaller than, greater than or equal to v_p, due to this relation.

Dispersive Medium, in dispersive medium, all wave have different velocity. (if we incident ray in dispersive medium then it will disperse in 7-colours)

$$\frac{dv_p}{d\lambda} \neq 0 \quad (\& \; v_p = \nu \lambda)$$

Non-Dispersive medium, All wave travell with same velocity

$$\frac{dv_p}{d\lambda} = 0$$

$$\boxed{v_g = v_p}$$

Despersive Medium

Normal

$\frac{dv_p}{d\lambda} = +ve$

$$v_g = v_p - \lambda \frac{dv_p}{d\lambda}$$

if $\frac{dv_p}{d\lambda} = +ve$

then $\boxed{v_g < v_p}$

Anamalous

$\frac{dv_p}{d\lambda} = -ve$

$$v_g = v_p - \lambda \frac{dv_p}{d\lambda}$$

$$\boxed{v_g > v_p}$$

In terms of Refractive Index :-

$$\boxed{n = \frac{c}{v}}$$

Normal $\rightarrow$ $\frac{dn}{d\lambda} = -ve$

Anamoly $\rightarrow$ $\frac{dn}{d\lambda} = +ve$

from $\omega \propto k$, we can find medium is either dispersive or Non-dispersive

As $\omega = Ak$. $\frac{\omega}{k} = v_g = A$ $\boxed{\frac{dv_g}{d\lambda} = 0}$

- If $\boxed{\omega \propto k} \rightarrow$ Non-dispersive
- ω Not proportional to k i.e. ω is not linearly varries with k then dispersive.

Proof :- Group velocity is equal to the classical velocity of the particle.

$$v_{classical} = \frac{p}{m}$$

i.e. $v_g = \frac{p}{m}$ (To show)

$$E = \frac{p^2}{2m} + V(x)$$

$$v_g = \frac{dE}{dp} = \frac{2p}{2m} = \frac{p}{m}$$

$$\boxed{v_g = \frac{p}{m}}$$

Centre of wave packet show the classical behaviour.

So wave packet show the connection b/w classical & Quantum mech.

Que :- Show that the phase velocity of De Broglie wave associated with a particle moving with relavistic speed (v) is greater than speed of light.

$v_p > c$

Particle velocity $= v$

$v_g = v$ (particle's vel. is the interprete as group vel.)

$\lambda = \frac{h}{m v_g} \Rightarrow k = \frac{2\pi}{\lambda}$

$\nu = \frac{E}{h} = \frac{mc^2}{h} \Rightarrow \omega = \frac{E}{\hbar}$

$v_p = \nu \lambda$

$= \frac{mc^2}{h} \cdot \frac{h}{m v_g}$

$v_p = \frac{c^2}{v_g} = c \cdot \frac{c}{v_g}$ or $\boxed{v_p v_g = c^2}$

$\frac{c}{v_g} >> 1$ for any material particle $v < c$

$\therefore \boxed{v_p > c}$

Que :- The dispersion relation of a certain wave is

$\omega = \sqrt{c^2 k^2 + m_0^2}$

where $\omega \rightarrow$ angular freq. , $k \rightarrow$ wave vector , $c \rightarrow$ speed of light & $m_0 \rightarrow$ any constant.

The group velocity v has the following propert

(i) $v \rightarrow c$ as $k \rightarrow 0$ & $v \rightarrow c$ as $k \rightarrow \infty$

(ii) $v \rightarrow c$ as $k \rightarrow 0$ & $v \rightarrow \infty$ as $k \rightarrow \infty$

(iii) $v \rightarrow 0$ as $k \rightarrow 0$ & $v \rightarrow \infty$ as $k \rightarrow \infty$

(iv) $v \rightarrow 0$ as $k \rightarrow 0$ & $v \rightarrow c$ as $k \rightarrow \infty$

Date :

Page No. :

$V_g = v \quad , \quad \omega = \sqrt{c^2k^2 + m_0^2}$

$$V_g = v = \frac{d\omega}{dk} = \frac{1}{2}\frac{2kc^2}{\sqrt{c^2k^2+m_0^2}}$$

$$\therefore v = \frac{c^2}{\sqrt{c^2 + \frac{m_0^2}{k^2}}}$$

If $k = 0$ then $v \Rightarrow 0$

$k = \infty$ then $v = c$

(d) → correct

Qu: In a certain medium the wave no. k & freq. ω are related by the relation,

$$\omega^2 = c^2k^2(1+ak^2)$$

where c & a are constants if V_g is the group velocity & V_p is the phase velocity then

(a) $V_g = V_p$

(b) $\frac{V_p}{V_g} = \frac{(1+ak^2)}{(1+2ak^2)}$

(c) $V_g V_p = c^2$

(d) $V_g V_p = c^2(1+2ak^2)$

$$\omega = ck\sqrt{1+ak^2}$$

$$V_g = \frac{d\omega}{dk}$$

$$= ck \cdot \frac{1}{2}\frac{2ak}{\sqrt{1+ak^2}} + c\sqrt{1+ak^2}$$

$$= \frac{ack^2}{\sqrt{1+ak^2}} + c\sqrt{1+ak^2}$$

$$= \frac{ack^2 + c(1+ak^2)}{\sqrt{1+ak^2}} = \frac{c+2ack^2}{\sqrt{1+ak^2}}$$

$$\frac{\omega}{k} = c\sqrt{1+ak^2} \Rightarrow v_p$$

So $\boxed{v_p v_g = c^2(1+2ak^2)}$

Or

$$2\omega \frac{d\omega}{dk} = c^2\left[k^2(2ka) + (1+ak^2)2k\right]$$

$$2\frac{\omega}{k}\frac{d\omega}{dk} = \frac{c^2}{k}\left[2k^3a + 2k + 2ak^3\right].$$

$$2 v_p v_g = \frac{c^2}{k}\left[4k^3a + 2k\right]$$

$$2 v_p v_g = c^2\left[4k^2a + 2\right]$$

$$2 v_p v_g = 2c^2(2k^2a+1)$$

$$\Rightarrow \boxed{v_p v_g = c^2(1+2k^2a)} \text{ proved}$$

Q: The relation b/w angular freq. ω & wave no. k for given type of wave $\omega^2 = \alpha k + \beta k^3$

The wave no. k_0 for which the phase velocity = group velocity is

(a) $3\sqrt{\frac{\alpha}{\beta}}$ (b) $\frac{1}{3}\sqrt{\frac{\alpha}{\beta}}$ ✓(c) $\sqrt{\frac{\alpha}{\beta}}$ (d) $\frac{1}{2}\sqrt{\frac{\alpha}{\beta}}$

$$v_p = v_g.$$

$$\omega^2 = \alpha k + \beta k^3$$

$$2\omega\frac{d\omega}{dk} = \alpha + 3\beta k_0^2$$

$$2\frac{\omega}{k}\frac{d\omega}{dk} = \frac{\alpha}{k_0} + 3\beta k_0$$

$$v_p v_g \cancel{v_p} = \frac{1}{2}\frac{\alpha}{k_0} + \frac{3}{2}\beta k_0$$

Date :
Page No. :

$$v_p^2 = \cancel{\frac{\omega^2}{k^2}} \cancel{\Rightarrow} \frac{1}{2}\frac{\alpha}{k_0} + \frac{3}{2}\beta k_0$$

$$v_p^2 = \frac{\omega^2}{k^2} \cancel{\Rightarrow v_p =}$$

$$\frac{\omega^2}{k^2} = \frac{1}{2}\frac{\alpha}{k_0} + \frac{3}{2}\beta k_0$$

$$\frac{\alpha}{k_0} + \beta k_0 = \frac{1}{2}\frac{\alpha}{k_0} + \frac{3}{2}\beta k_0$$

$$\alpha + \beta k_0^2 = \frac{\alpha}{2} + \frac{3}{2}\beta k_0^2$$

$$\alpha - \frac{\alpha}{2} = \beta k_0^2\left(-1 + \frac{3}{2}\right) = \beta k^2 \frac{1}{2}$$

$$\frac{\alpha}{\cancel{2}} = \beta k_0^2 \frac{1}{\cancel{2}} \Rightarrow k_0^2 = \frac{\alpha}{\beta}$$

$$\boxed{k_0 = \sqrt{\frac{\alpha}{\beta}}}$$ (c) → correct

Qus:- The ang. freq. of the surface wave in a liquid is given by $\omega = \sqrt{gk + \frac{Tk^3}{\rho}}$

If g = acceleration due to gravity
ρ → density of liquid
T → Surface tension
find the phase & group velocity if
1. surface wave have very large w.L.
2. surface " " small "

$$\omega = \sqrt{gk + \frac{Tk^3}{\rho}} \qquad k = \frac{2\pi}{\lambda}$$

$$\omega = \sqrt{\frac{2\pi g}{\lambda} + \frac{T}{\rho}\left(\frac{2\pi}{\lambda}\right)^3}$$

Case I :- λ → large so $\frac{1}{\lambda}$ → small
so II term neglect w.r.to I term

$$\omega = \sqrt{gk}$$

Case 2 :- $\lambda \rightarrow$ small, $\frac{1}{\lambda} \rightarrow$ large so its higher powers are more large.

So neglect I term w.r.to II.

So $$\omega = \sqrt{\frac{TK^3}{\rho}} = \sqrt{\frac{T}{\rho}\left(\frac{2\pi}{\lambda}\right)^3}$$

Now v_g & v_p = ?

for Case 1 :- $$v_p = \frac{\omega}{K}$$

$$= \frac{\sqrt{gK}}{K} = \sqrt{\frac{g}{K}} = \sqrt{\frac{g}{2\pi/\lambda}}$$

$$v_p = \sqrt{\frac{g\lambda}{2\pi}}$$

$$v_g = \frac{d\omega}{dK} = \sqrt{g}\,\frac{1}{2}\frac{1}{\sqrt{K}} = \frac{1}{2}\sqrt{\frac{g}{K}} = \frac{1}{2}\sqrt{\frac{\lambda g}{2\pi}}$$

$$v_g = \frac{1}{2}\sqrt{\frac{g\lambda}{2\pi}} = \frac{v_p}{2}$$

for Case 2 :- $$v_p = \frac{\omega}{K} = \frac{1}{K}\sqrt{\frac{TK^3}{\rho}} = \sqrt{\frac{TK}{\rho}}$$

$$v_p = \sqrt{\frac{2\pi T}{\rho\lambda}}$$

$$v_g = \frac{d\omega}{dK} = \sqrt{\frac{T}{\rho}}\,\frac{3}{2}\sqrt{K} = \frac{3}{2}\sqrt{\frac{TK}{\rho}}$$

$$v_g = \frac{3}{2}v_p$$

Date :
Page No. :

Heisenberg Uncertainity Principle :-

In Classical mech., particle's position, mom., energy etc. can be calculated simultaneously bc to C.M. particle behave like particle.

Acc. to Q. Mech., If we know the position but ~~the corresponding momentum will be uncertain.~~ So in Q.M. we can not calculate the position & mom. simultaneously with accuracy

$$\boxed{\Delta x \, \Delta p_x \geq \frac{\hbar}{2}}$$

$$\Delta x \cdot \Delta K \geq \frac{\hbar}{2}$$

$$\Delta E \cdot \Delta t \geq \frac{\hbar}{2}$$

For Canonicaly conjugate ~~classical pair~~ dynamical vari pair (the unit of the product of position & mom will be unit of action)

& pair which satisfies the Canonical eqn of motion $\dot{q} = \frac{\partial H}{\partial p}$ & $\dot{p} = \frac{-\partial H}{\partial q}$

will be conjugate variable. Uncertainity principle is valid only for Conjugate dynamical vari

Position space wave funcn & mom. sp Wave funcn are fourier transform of each other i.e. we can convert position space Wave funcn into mom. space wave funcn or vice versa.

If two operator commute then we get ~~uncertainity~~ ~~reln~~. Uncertainity product = 0 i.e. we can measure posi mom. simultan

$$[A, B] = 0 \Rightarrow AB = BA$$

$$[x, p'_x] = 0$$

$$\Delta x \cdot \Delta p_x \geq \hbar/2$$

$$\Delta x \cdot \Delta p' = 0$$

Applications

(1) Non Existance of e^- inside the Nucleus :-
(2) To calculate the Bohr radius
(3) To calculate the ground state energy

(1) Non Existance of e^- inside the nucleus :-

Suppose e^- exist inside the nucleus then uncertai will be of the order of size of nucleus

So uncertainity in position

$$\Delta r = 2 \times 10^{-15} \quad \text{(Radius of nucleus)}$$

$$\Delta r \cdot \Delta p \geq \frac{\hbar}{2}$$

$$\Delta p \geq \frac{\hbar}{2\Delta r}$$

$P \to P - \Delta P$ to $P + \Delta p$:

$$p \sim \Delta p$$

$$K.E. = \frac{p^2}{2m} = \frac{(\Delta p)^2}{2m} = \frac{1}{2m}\left(\frac{\hbar}{2\Delta r}\right)^2 = 97 \text{ or } 96 \text{ MeV}$$

If e^- exist in the nucleus then K.E. of e^- is 97 or 96 MeV acc. to Uncertainity principle But the experimental value of K.E. of e^- is 4 MeV.

There is a large difference in K.E. So e^- can not exist inside the nucleus.

(II) & (III) Bohr radius & ~~ground state~~ ~~properties~~ Energy :-

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

Min uncertainity $\Delta x \; \Delta p = \frac{\hbar}{2}$

$$\text{total energy} = \frac{p^2}{2m} + V$$

Date :
Page No. :

$p \to \Delta p$ If V is fuc^n of x or (r in 3 dim) then
$x \sim \Delta x$

hence energy will be the fuc^n of Δx,

$$E(\Delta x) = \frac{p^2}{2m} + V$$

$$\frac{dE}{d(\Delta x)} = 0$$

$$E(\Delta x) = \frac{(\Delta p)^2}{2m} + V(x) \qquad \Delta x \Delta p = \frac{\hbar}{2}$$

$$= \frac{(2\Delta x)^2}{2m} + V(\Delta x) \qquad \Rightarrow \Delta p = \frac{\hbar}{2\Delta x}$$

$$E = \frac{2(\Delta x)^2}{m} + V(\Delta x)$$

$$\frac{dE}{d(\Delta x)} = 0 \Rightarrow \frac{2}{m} \cdot 2\Delta x \cancel{=0} + \frac{dV}{d(\Delta x)} = 0$$

Que :- Calculate the ground state energy for a linear harmo oscillator by using Heisenberg uncertainity principle

for linear Harmonic oscillator,

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \qquad k = m\omega^2$$

Let $p \sim \Delta p$

$x \sim \Delta x$

$$\therefore E \sim \frac{(\Delta p)^2}{2m} + \frac{1}{2}m\omega^2(\Delta x)^2$$

Now, using Heisenberg Uncertainity principle

$$\Delta x . \Delta p \geqslant \frac{\hbar}{2}$$

If the fucn is of Gaussian type then

$$\Delta x \, \Delta p = \frac{\hbar}{2}$$

{ Gaussian type fucn is of the form
$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\Delta x)^2}{2\sigma^2}}$$ }

for Non-Gausian type fucn,

$$\Delta x \, \Delta p > \frac{\hbar}{2} \text{ always}$$

All Other fucn (other than ground state of harmonic oscillator) fu is of Non-Gaussian type (if not given in Ques

$$\therefore E = \frac{\left(\frac{\hbar}{2\Delta x}\right)^2}{2m} + \frac{1}{2}m\omega^2(\Delta x)^2$$

$$E = \frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2}m\omega^2(\Delta x)^2$$

Now Put $\frac{dE}{d(\Delta x)} = 0$ & calculate Δx

Date :
Page No. :

$$\frac{dE}{d(\Delta x)} = \frac{\hbar^2}{8m}\left(\frac{-2}{(\Delta x)^3}\right) + m\omega^2(\Delta x) = 0$$

$$\frac{-\hbar^2}{4m(\Delta x)^3} + m\omega^2(\Delta x) = 0$$

$$+\frac{\hbar^2}{4m(\Delta x)^3} = m\omega^2(\Delta x)$$

$$(\Delta x)^4 = \frac{\hbar^2}{4m^2\omega^2}$$

$$\boxed{\Delta x = \left(\frac{\hbar}{2m\omega}\right)^{1/2}}$$

Now put this Δx in energy eqn then we get the ground state energy.

$$E = \frac{\hbar^2}{8m}\frac{2m\omega}{\hbar} + \frac{1}{2}m\omega^2 \cdot \frac{\hbar}{2m\omega}$$

$$E = \frac{\hbar\omega}{4} + \frac{\omega\hbar}{4}$$

$$\boxed{E = \frac{1}{2}\hbar\omega}$$

Qn :- Using the Heisenberg Uncertainity principle. Calc the min. possible energy of a particle of mass m, moving in potn

$$V(x) = \frac{V_0}{a}|x|$$

$$E = \frac{p^2}{2m} + \frac{V_0}{a}|x|$$

$\Delta x \Delta p \simeq \hbar$

$p \sim \Delta p$

$x \sim \Delta x$

$$E = \frac{(\Delta p)^2}{2m} + \frac{V_0}{a}\Delta x$$

$$E = \frac{\hbar^2}{2m(\Delta x)^2} + \frac{V_0}{a}\Delta x$$

$$\frac{dE}{d(\Delta x)} = 0$$

$$\frac{\hbar^2(-2)}{2m(\Delta x)^3} + \frac{V_0}{a} = 0$$

$$\frac{\hbar^2}{m} = \frac{V_0}{a}(\Delta x)^3$$

$$\Rightarrow \quad \Delta x = \left(\frac{\hbar^2 a}{V_0 m}\right)^{1/3}$$

$$E = \frac{\hbar^2}{2m\left(\frac{\hbar^2 a}{V_0 m}\right)^{2/3}} + \frac{V_0}{m}\left(\frac{\hbar^2 a}{V_0 m}\right)^{1/3}$$

$$E = \frac{\hbar^2 + 2V_0\left(\frac{\hbar^2 a}{V_0 m}\right)}{2m\left(\frac{\hbar^2 a}{V_0 m}\right)^{2/3}}$$

$$E = \frac{\hbar^2 + \frac{2\hbar^2 a}{m}}{2m\left(\frac{\hbar^2 a}{V_0 m}\right)^{2/3}}$$

$$\boxed{E = \frac{3}{2}\left(\frac{\hbar^2 V_0^2}{m a^2}\right)^{1/3}}$$

12-8-2012

Que: Calculate the ground state energy of Hydrogen atom by using Heisenberg Uncertainity principle.

Hamiltonian for

$$H = \frac{p^2}{2m} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r} \quad (M.K.S.)$$

$$H = \frac{p^2}{2m} - \frac{e^2}{r} \quad (C.G.S.)$$

When $p \sim \Delta p$

$x \sim \Delta x$

Gaussian wave packet is called min. uncertainty wave packet.

Date :

Page No. :

$$E = \frac{(\Delta p)^2}{2m} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{\Delta r}$$

By Uncertainity principle $\Delta r \cdot \Delta p \approx \hbar$

$\Delta p \approx \frac{\hbar}{\Delta r}$

$$E = \frac{\hbar^2}{2m(\Delta r)^2} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{\Delta r}$$

$$\frac{dE}{d(\Delta r)} = 0$$

$$\frac{dE}{d(\Delta r)} = \frac{\hbar^2}{\cancel{2}m}\left(\frac{-\cancel{2}}{(\Delta r)^3}\right) - \frac{e^2}{4\pi\epsilon_0}\left(\frac{-}{\Delta r^2}\right) = 0$$

$$\Rightarrow \frac{\hbar^2}{m(\Delta r)^3} = \frac{e^2}{4\pi\epsilon_0(\Delta r)^2}$$

$$\Delta r = \frac{\hbar^2}{m}\frac{4\pi\epsilon_0}{e^2}$$

$$\Delta r = \frac{4\pi\epsilon_0\hbar^2}{me^2}$$

The value of Δr is called the radius of Ist Bohr orbit,

Put Δr in energy expression, then we obtain the ground state energy.

$$E = \frac{\hbar^2}{2m\left(\frac{4\pi\epsilon_0\hbar^2}{me^2}\right)^2} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{\left(\frac{4\pi\epsilon_0\hbar^2}{me^2}\right)}$$

$$= \frac{me^4\hbar^2}{32\pi^2\epsilon_0^2\hbar^4} - \frac{me^4}{16\pi^2\epsilon_0^2\hbar^2}$$

$$= \frac{me^4}{\hbar^2(4\pi\epsilon_0)^2}\left[\frac{1}{2} - 1\right]$$

$$\boxed{E = -\frac{me^4}{2\hbar^2(4\pi\epsilon_0)^2}}$$

another type of

in H_2 atom - 1p & 1e. If in case of H_2 atom which we assume the mass of p is lighter than mass of e^- then mass will be replaced by reduced mass.

$$E = -\frac{\mu e^4}{2\hbar^2 (4\pi\epsilon_0)^2} = -13.6\,eV$$

Wavefunction :- Classically, We can calculate any informat' by position vector (r).

$$v = \frac{dr}{dt}, \quad a = \frac{d^2r}{dt^2}, \quad p = mv, \quad K.E. = \frac{p^2}{2m}$$

$$F = ma \; ; \; v = -\int F \cdot dr$$

Similarly in Q.M., We can calculate all the information by the wave func^n.

The func^n that rep^n the state of Quantum mechanical system & contain all the information regarding that system. & represented by $\psi(x,t)$

Physical significance of ψ :- ψ itself has no physical meaning but ψ^2 has physical meaning & we can interprete $|\psi|^2$ as probability density.

$|\Psi(x,t)|^2 \Rightarrow$ This term rep^n position probability density

OR the probability of finding the particle at a particular point at a particular time t.

If we multiply this by position element (it is different in 1, 2, 3 dim.)

In 1D. $|\Psi(x,t)|^2 dx \rightarrow$ prob. of finding the particle in a region of space that lies b/w x & $x+$

Date :
Page No. :

In 3 dim, $|\psi(r,t)|^2 d\tau \Rightarrow$ The prob. of finding the partic[le in] a region of volum $d\tau$ that lies from r [to] $r+dr$.

$\int_{x_1}^{x_2} |\psi(x,t)|^2 dx \Rightarrow$ The prob. of finding the pa[rticle] ~~in a region of space lies b/w~~

But here the region of space is ~~small~~ large. & In $|\psi(r,t)|^2 d\tau$ region of space is small.

For small region, just multiply ~~by~~ d[ensity] by small distance.

In large region, integrate b/w 2 point[s]

$\boxed{|\psi|^2 = \psi^* \psi}$

Normalization Condition :-

$$\int_{\text{all space}} \psi^* \psi \, d\tau = 1$$

This is normalized to unity. This term re[presents] the exact prob. term.

It is valid for localized wave fun^n for w[hich] it gives unity normalization bu[t] for non-lo[calized] wave fun^n it gives value $\neq 1$.

Phy. Meaning :- The prob. of finding the particle in all sp[ace] is 1.

Orthonormality :- If more than 1 states are pre[sent] at a time then resultant wave fun^n will be linear combination of wave fun^{n}s of all the stat[es].

for Harmonic Oscillator $\psi = c_0 \psi_0 + c_1 \psi_1$

Here 2 states are present. ψ is linear combina[tion] of ψ_0 & ψ_1.

Then there will be the orthogonality te[rm] Cond^n $\int \phi^* \phi \, d\tau = 0$

Orthonormality Condⁿ → & $\boxed{\int_{\text{all space}} \phi_i^* \phi_j \, d\tau = \delta_{ij}}$ $= 1 , \; i = j$
$= 0 \quad i \neq j$

Orthonormality is a combination of Orthogonality & Normalization.

If $\psi = C_0 \psi_0 + C_1 \psi_1$

$$\psi^* \psi = (C_0^* \psi_0^* + C_1^* \psi_1^*)(C_0 \psi_0 + C_1 \psi_1)$$

$$= |C_0|^2 \psi_0^* \psi_0 + |C_1|^2 \psi_1^* \psi_1 + \underbrace{C_0^* C_1 \psi_0^* \psi_1 + C_1^* C_0 \psi_1^* \psi_0}_{= 0}$$

$$= |C_0|^2 \psi_0^* \psi_0 + |C_1|^2 \psi_1^* \psi_1$$

{ ~~In cas~~ By orthogonali

If 2 ~~p~~ states are possible then particle can be i ψ_0 state or in ψ_1 state. It is not confirmed that particle is in ground state or in excited state. There is 50% - 50% prob. ~~of~~ for particle to b in both states. It may be in both states.

So We use Orthonormality in which we ta the linear combination of these 2 ~~waves~~ states -

$$\psi = \frac{1}{\sqrt{2}} \psi_0 + \frac{1}{\sqrt{2}} \psi_1$$

Probability is additive so

$$\boxed{P = P_1 + P_2}$$

If both indices are same, then condⁿ of ~~Orthog~~ orthonormality is converted into ~~ort~~ Normaliza

If both indices are diffrent, then condⁿ of orthonormality is converted into Orthogonality

$$\int \phi_i^* \phi_j \, d\tau = 0 \Rightarrow \text{Cond}^n \text{ of Orthogonality.}$$

The prob of finding the particle in more than 2 states is zero

Date :
Page No. :

Condition of physically acceptable Wave fuc^n / Well behaved Wave fuc^n OR Physically admissible wave fuc^{n}s :-

Conditions :- (1) ψ must be finite.

~~(2) ψ must be single valued & c~~

(3) Ist derivative of ψ i.e. $\frac{d\psi}{dx}, \frac{d\psi}{dy}, \frac{d\psi}{dz}, \frac{d\psi}{dt}$ be finite & single valued.

(4) Ist derivative of ψ i.e. $\frac{d\psi}{dx}, \frac{d\psi}{dy}, \frac{d\psi}{dz}$ mus continuous.

(Continuity on $\frac{d\psi}{dt}$ is not cheaked)

(5) ψ must be square integrable.

$$\int_{\text{all space}} \psi^* \psi \, d\tau = \text{finite}$$

~~fuc^n rep^n the~~ It depand on wave fuc^n ψ.

If ψ is localised $\rightarrow$ It rep^n particle

If ψ is Non-localised $\rightarrow$ It rep^n wave

depand upon localised or Non-localised

If these cond^{n}s are satisfied then wave fuc^n is acceptable.

$\Rightarrow$ For finite prob., wave fuc^n must be finite

$P = |\psi|^2 = \psi^* \psi$ if $\psi = \infty$ then $P = \infty$

$\Rightarrow$ $|\psi|^2 = \frac{1}{2}, \frac{1}{4}$ then there will be more then prob. at a time but prob. car have a single value at a time.

$\therefore$ So ψ must be single valued

$\Rightarrow$ If ψ is discontinuous then particle can not d motion. So ψ should be continous.

$\Rightarrow$ Mom. operator $P = -i\hbar \nabla$

Regarding the system for calculating mom

We have to find ~~&~~ $-i\hbar \frac{d\psi}{dx}$, $-i\hbar \frac{d\psi}{dy}$, $-i\hbar \frac{d\psi}{dz}$

Mom. should be finite. Mom will be finite only when these derivatives are finite.

⇒ At a time mom. & velocity are single value so derivatives are single valued.

⇒ $E\psi = i\hbar \frac{d\psi}{dt}$

Energy will be finite only if $\frac{d\psi}{dt}$ is finite

b.oz $\underbrace{E\psi = \lambda\psi}$ must be finite

⇒ We can calculate the derivative only if the fu is continous.

To calculate K.E., we have to calculate II order derivative

$$\frac{p^2}{2m} = -\frac{\hbar^2}{2m}\nabla^2$$

$$\nabla^2 = \frac{d^2\psi}{dx^2} = \frac{d}{dx}\left(\frac{d\psi}{dx}\right)$$

We can calculate IInd order derivative only if I order derivative is continous.

No physical operator is possible which can find IInd order derivative of fucn w.r. to time

⇒

Inspite of being ~~of~~ ψ finite; Integral of ψ may be infinite but we take integral over all spac it'll have finite value.

Note :- If $V = \infty$ then $\frac{d\psi}{dx}$, $\frac{d\psi}{dy}$, $\frac{d\psi}{dz}$ are discontinu

We can not cheak continuty if potn is ∞.

~~If $V = 0$ then derivatives must be continuous.~~

Date :
Page No. :

How to cheak the Cond^n's :- $\psi(x,t)$

(1) $x \rightarrow$ from $-\infty$ to $+\infty$.

Wave fuc^n should by finite in all region from to $+\infty$.

ψ = finite

(2) $\psi(x,t) \Rightarrow \psi(x)$

for ~~no~~ single value of independent variable (x); more than 1 value of ψ should not present.

If $\psi = x^2$

$\psi = \pm\sqrt{x}$

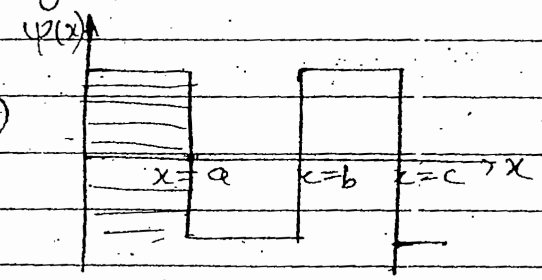
for 1 value of x.

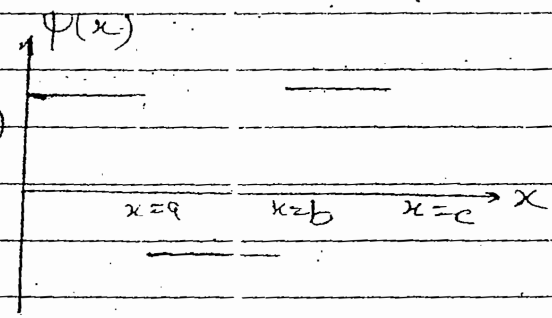
Here 2 values of ψ are present, so it is not ~~continous~~ single valued.

(3) If fuc^n is given then mathematical form is

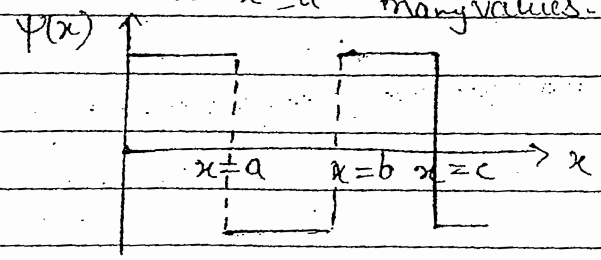
⇒ If L.H.L = R.H.L. = f(a) at $x = a$ then fuc^n continous.

If plot is given then, there should ~~be~~ not any break.

(1)


(2)


at $x = a$ many values.

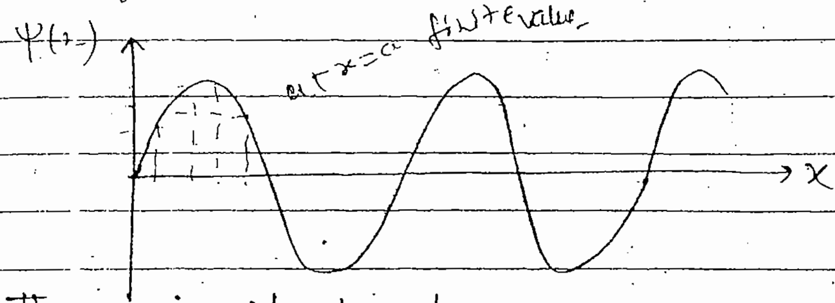
(3)


All three plots are discontinous.

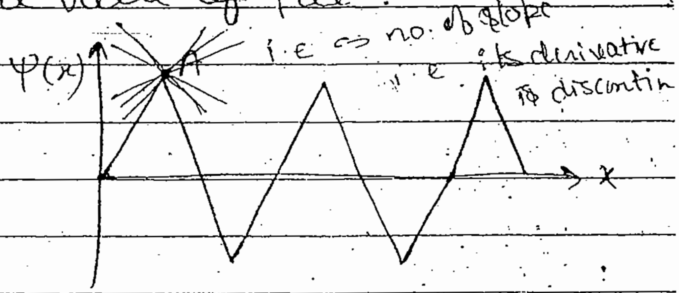
At $x = a$, we can not define the value of fuc^n for all 3 plots.

If at 1 points, ∞ no. of values are possib

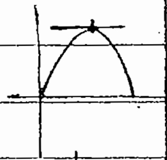
Of a fucⁿ then fucⁿ is discontinous at that point.



There is No break.
At each points, we get finite value of fucⁿ. i.e at 1 point, we get only one value of fucⁿ.



This fucⁿ is continous
But derivative is (i.e slop) discontinous.
If we draw tangent at A then at A we can draw ∞ no. of tangents so there may be ∞ no. of slops, $\frac{d\psi}{dx}$ then there will be ∞ no. of values of derivatives.
These fucⁿ is single valued but its derivative have ∞ no. of values
So its derivative is discont. but fucⁿ is Cont.



Prob :- Among the following wave fucⁿs which repⁿ the physically acceptable wave fucⁿ.

(1) $3\sin\pi x$ (2) $4-|x|$
(3) $\pm\sqrt{5x}$ (4) $e^{-\lambda x}$

~~(1) : $3\sin\pi x$~~
~~$x \to -\infty$ to $+\infty$~~

(4) $e^{-\lambda x}$

if $x \to +\infty$ then $e^{-\infty} = 0$

from 0 to ∞ in all space it is finite ~~but~~ for +ve side

If fun^n is finite then square integral must be checked

Date :
Page No. :

$x \to -\infty$

$e^{-\lambda x} = e^{\infty} = \infty$

for $-$ve side it is not acceptable

if we check further,

~~for~~ single value of x, $e^{-\lambda x}$ gives ~~single value.~~

Square integral $\int \psi^* \psi \, d\tau = 1$

$$\int_{-\infty}^{\infty} e^{-2\lambda x} dx = \underbrace{\int_{-\infty}^{0}}_{\infty} + \underbrace{\int_{0}^{\infty}}_{fix}$$

$= \infty$

So (4) $e^{-\lambda x}$ is not acceptable (not finite & sq.

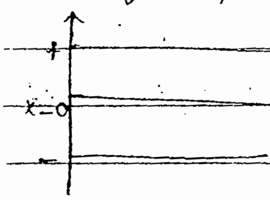
(3) $f = \pm\sqrt{5x}$

→ It is not finite, for $x = \infty$, $f = \infty$

→ for single value of x, f give double value.

→ Continous cond^n :- At $x=0$, f is defined.

So it is continous.



→ Square integral :-

$$\int_{-\infty}^{\infty} \pm\sqrt{5x}\, dx = \infty$$

Not satisfied.

(3) $\pm\sqrt{5x}$ Not acceptable.

(2) $f = 4 - |x|$

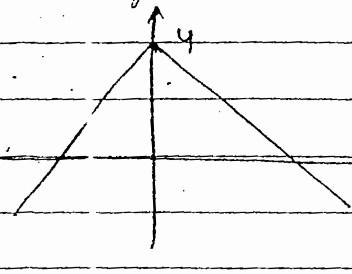
→ $x = \infty$, $f = \infty$

→ for $x \to$ single valued, f is single value

→ At $x=0$, f is finite

$x=0$, $f=4$

Continous.



→ $\frac{df}{dx} = -1$ for +ve side $f = 4-x$

for $-$ve side, $f = 4+x$, $\frac{df}{dx} = 1$

A- $x=0$, fuc^n is single valued but its derivative has ∞ no. of value.

So $\frac{df}{dx}$ is discontinuous.

* Not acceptable

(1) $\psi = 3\sin\underbrace{\pi x}_{\theta}$

→ for $x = +\infty$ or $-\infty$ ⇒ $\sin\theta = -1$ to 1, $\psi \to$ finite.

→ for single value of θ, ψ will be single valued.

→ Continuous :-

It is cont^n for all value of x.

→ $\frac{d\psi}{dx} = 3\pi\cos\pi x$

at $x = \pm\infty$, $\cos\theta = -1$ to 1

So $\frac{d\psi}{dx}$ = finite, single valued, continuous

→ Square integral :- $\int_{-\infty}^{\infty} \cancel{B}\,\psi^*\psi\, d\tau \neq 1$

$= \int_{-\infty}^{\infty} 3\sin\pi x \; 3\sin\pi x \, dx$

$= 9\int_{-\infty}^{\infty} \sin^2\pi x\, dx = 9\int_{-\infty}^{\infty}\left[\frac{1-\cos 2\pi x}{2}\right]dx$

$= \infty$

Not satisfied

{ I term $\int_{-\infty}^{\infty} 1\, dx = [x]_{-\infty}^{\infty} =$

Not acceptable.

It is acceptable for limited space.

More appropriate answer is Option (1).

Date :
Page No. :

Que 1- Which of the following wave fun^n rep^n acceptable wave fun^n of the particle in the range $-\infty \le x$:

(a) $\phi(x) = A \tan x$, $A > 0$

(b) $\phi(x) = B \cos x$, $B =$ real

(c) $\phi(x) = c\, \exp\left(\frac{-D}{x^2}\right)$, $c > 0, D < 0$

(d) $\phi(x) = E x e^{-Fx^2}$, $E, F > 0$

(a) $\phi(x) = A \tan x$

$\Rightarrow x = \pm\infty$, $\phi(x) = \cancel{A} \frac{\sin\infty}{\cos\infty} \Rightarrow$ Not finite.

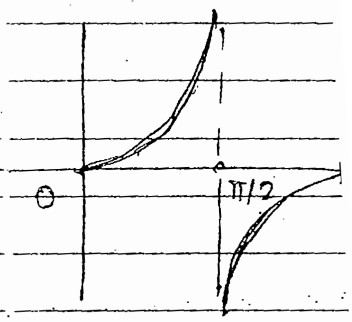
$x = \frac{\pi}{2}$, $\phi(x) = \infty$

$\rightarrow$ for single value of x, $\phi(x)$ is single valued.

$\rightarrow$ Continous :- At $\frac{\pi}{2}$, continuty

$\rightarrow$ break. So At $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$

$\phi(x)$ is discontinous.



Not acceptable

(b) $\phi(x) = B \cos x$

$\rightarrow$ for $x = -\infty$ to $+\infty$, $\phi(x) \rightarrow$ finite

$\rightarrow$ Single valued

$\rightarrow \phi(x) = B\cos x$

It is continous

$\rightarrow \frac{d\phi}{dx} = -B \sin x$

$\rightarrow$ sq. Integral $= |B|^2 \int_{\infty}^{\infty} \cos^2 x \, dx = \infty$

Not acceptable

Date :
Page No. :

(3) $\phi(x) = c\exp\left(\frac{-D}{x^2}\right)$ ∵ $\phi(x) = c\exp\left(\frac{D}{x^2}\right)$

At $x = 0$, $\phi(x) = \infty$ bcoz $D<0$

in all space

Not acceptable.

(4) $\phi(x) = Ex\,e^{-Fx^2}$

This is the product of two func's ψ_1 & ψ_2.

$\psi_1 = x \rightarrow \psi_2 = Ee^{-Fx^2}$

At $x = \pm\infty$, $\psi_1 = \pm\infty$ & $\psi_2 = 0$

But for the product

$x = 0$,

$\phi(x) = 0 \times 1 = 0$

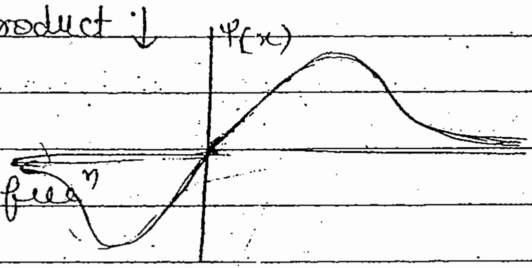
As $x\uparrow$, $\psi_1 \uparrow$ & $\psi_2 \downarrow$

$\psi_1 \rightarrow$ increasing func^n

$\psi_2 \rightarrow$ decreasing "

Slop of ψ_2 is larger than ψ_1 (slop of ψ_1 is constant) So Total product $\downarrow$

So $\psi = \psi_1\psi_2$



The slop of decreasing func^n is large.

So Total slop tends to 0.

for $x = -1$, $\psi_1 = +x = -1$

$\psi_2 = Ee^{-Fx^2} = Ee^{-F(-1)^2}$

ψ_1 & ψ_2 both are decreasing func's

So Product is $\downarrow$ upto some value & (then it starts $\uparrow$) ~~then~~ tend to zero.

Date:
Page No.:

* If there is addition of 2 fucⁿ then
$\psi = \psi_1 + \psi_2$
if ψ_1 & ψ_2 are continous then ψ will be conti

This is possible for addition, substraction, multipli
~~but Not for division.~~
Same Condⁿ for derivative.

So ψ is continous & $\frac{d\psi}{dx}$ is also continous.

Square Integral :- $\int_{-\infty}^{\infty} \psi^* \psi \, dx$

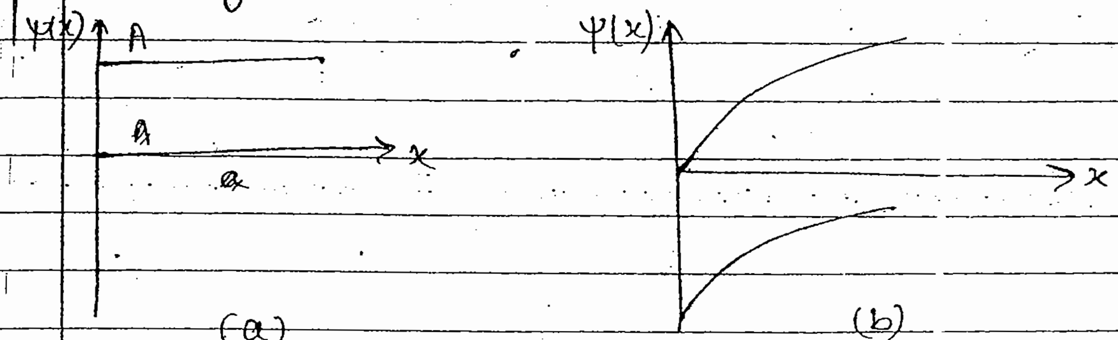
$|E|^2 \int_{-\infty}^{\infty} x^2 e^{-2fx^2} dx = 2E^2 \int_0^{\infty} x^2 e^{-2fx^2} d$

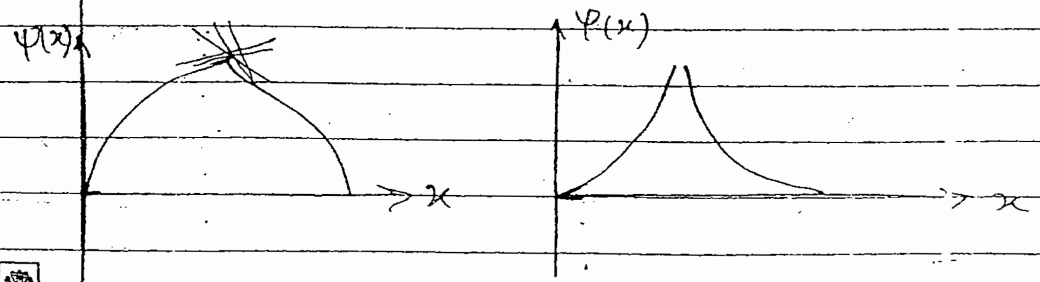
$= 2E^2 \int_0^{\infty} x^2 e^{-2fx^2} dx$

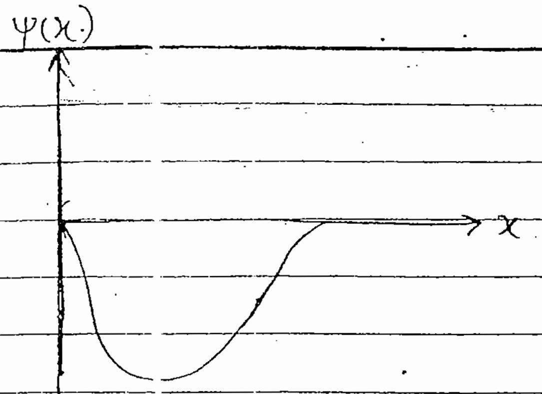
$=$ finite

Acceptable. $\left\{ \int_0^{\infty} e^{-x} x^{n-1} dx = \Gamma n \right\} n:$

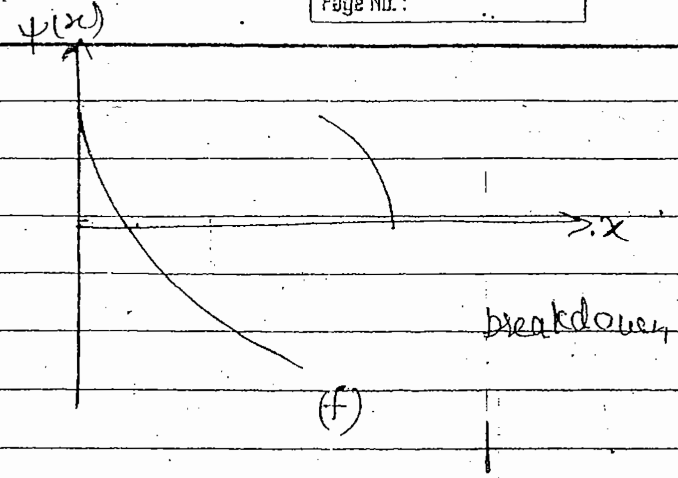
Prob:- Which of the wave fucⁿs shown in figure can n have physical significance in the interval shou & why ?







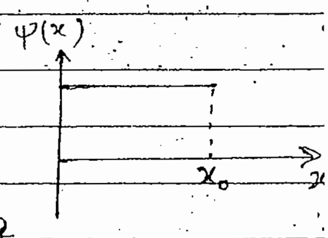
(e)



(f)

(a) $\psi(x)$ = finite
= Single valued
= Continuous

derivative $\frac{d\psi}{dx} = 0$ so finite, single valued & continous.



Square Integral :- $\int_0^{x_0} \psi^* \psi \, dx$

Integration b/w finite limits is finite

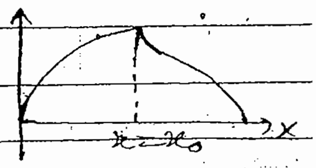
So fuc^n is physicaly acceptable & has physical significance.

(b) $\psi(x)$ = finite
= double valued

$\frac{d\psi}{dx}$ = finite, double valued

Square integral :- $\int \psi^* \psi \, dx$ = finite

So fuc^n has no physical signicance (it is multivalued)



(c) $\psi(x)$ = finite, single valued
= continous

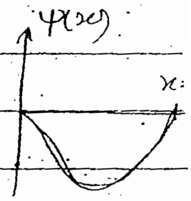
$\frac{d\psi}{\cdot}$ ⇒ at $x = x_0$. We can draw ∞ no.

So at $x = x_0$, $\frac{d\psi}{dx}$ is discontinous.

So fucⁿ has no physical significance.

(d) $\psi(x)$ = discontinous
fucⁿ has no physical significance.

(e) $\psi(x)$ = finite, single valued, continous
$\frac{d\psi}{dx}$ = " " "

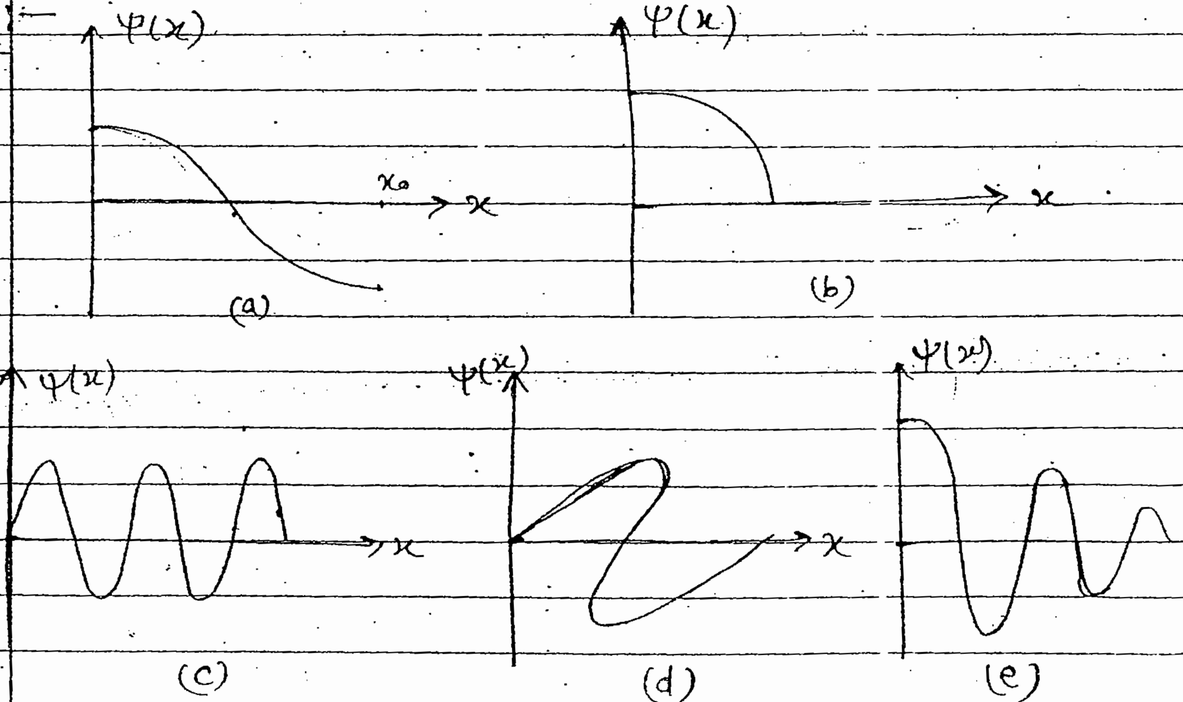


Square Integral $= \int_0^{x_0} \psi^* \psi \, dx$
$=$ finite

This fucⁿ has physical significance.

(f) $\psi(x)$ = discontinous, finite
fucⁿ has no physical significance.

Prob :-



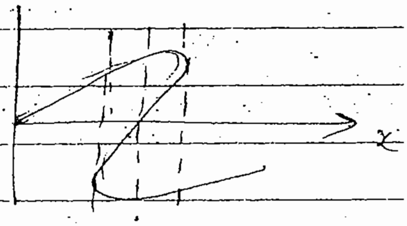
(a) $\psi(x)$ = finite, single valued, continous
$\frac{d\psi}{dx}$ = " " "

Square Integral = $\int_0^{x_0} \psi^* \psi \, dx$
cos type curve
$= \int_0^{x_0} \cos^2 x \, dx$ = finite

have physical significance.

(b) have physical signiface
(c) " " "

(d) $\psi(x)$ = Triple valued
(multivalued)
No physical signeficance



(e) $\psi(x)$ = finite, single valued, continous.
$\frac{d\psi}{dx}$ ⇒ amplitud is decreasing in this cur

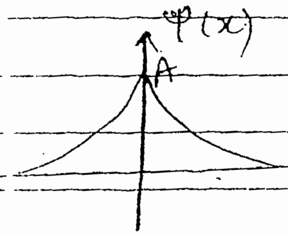
(i - amp. Not dec. then it is cos type curve)
So there is some decreasing func^n is multi-plied with cos func^n as $e^{-\lambda x} \cos x$.

$\frac{d\psi}{dx}$ = continous
at every point tangent is one.

Square integral :- $\psi(x) = Ae$
$\int_{-\infty}^{\infty} \psi^* \psi \, dx$ = finite

Physical significance

Que :- $\psi(x) = A e^{-\lambda|x|}$, $x \to -\infty$ to ∞



$\psi(x)$ = finite

at $x = \pm\infty$, $\psi(x) = 0$

$\psi(x)$ = single valued

= Continous

$\frac{d\psi}{dx}$ = ~~dis~~continous at $x = 0$

But $A^2 \int_{-\infty}^{\infty} e^{-\lambda|x|} e^{-\lambda|x|} dx = A^2 \int_{-\infty}^{\infty} e^{-2\lambda x} dx \Rightarrow$

for finite pot^n, it is not acc~~ptable~~ptable but dirac delta func^n it is acceptable.

$V(x) = -V_0 \delta(x-a)$

Note :- We can calculate the Normalization constant cond^n from

$\int \psi^* \psi \, d\tau = 1$

$\Rightarrow \psi^* \psi$ = unitless. We can calculate the unit only normalize wave func^n. ~~for localised~~

$|\psi|^2 = \frac{1}{d\tau}$

In 1D, $|\psi|^2 = \frac{1}{dx}$

$\psi = \sqrt{\frac{1}{dx}} = \sqrt{\frac{1}{\text{meter}}} = \frac{1}{\sqrt{\text{meter}}}$

In 2D, $|\psi|^2 = \frac{1}{d\tau} \Rightarrow \psi^2 = \frac{1}{L^2}$

$= \frac{1}{L}$ (unit of ψ)

In 3D, $|\psi|^2 = \frac{1}{L^3}$

$\psi = \frac{1}{L^{3/2}}$

Unit of Non-normalised func^n can not be defin

Q.1 :- If the probability that x lies b/w x & $x+dx$ is $p(x)\,dx = a e^{-ax}\,dx$ where $0<x<\infty$ & $a>0$.

Then the prob. that x lies b/w x_1 & x_2 where $(x_2>x_1)$ is

(a) $e^{-ax_1} - e^{-ax_2}$ (b) $a(e^{-ax_1} - e^{-ax_2})$

(c) $e^{-ax_2}(e^{-ax_1} - e^{-ax_2})$ (d) $e^{-ax_1}(e^{-ax_1} - e^{-ax_2})$

$$\int_{x_1}^{x_2} p(x)\,dx = \int_{x_1}^{x_2} a e^{-ax}\,dx$$

$$= \frac{a\left[e^{-ax}\right]_{x_1}^{x_2}}{-a} \cancel{(-a)} = -\left[e^{-ax_2} - e^{-ax_1}\right]$$

$$= e^{-ax_1} - e^{-ax_2}$$

(a) correct

Q.2 :- Consider a 1 Dim particle which is confined within the region $0 \le x \le a$ & whose wave fun^n in this region is $\psi(x,t) = \sin\left(\frac{\pi x}{a}\right) e^{-i\omega t}$. Calculate the probability of finding the particle in the interval $\frac{a}{4} \le x \le \frac{3a}{4}$

$$\int_{a/4}^{3a/4} |\psi(x,t)|^2\,dx = \int_{a/4}^{3a/4} \sin^2\left(\frac{\pi x}{a}\right) \quad \psi^* \psi\,dx$$

$$= \int_0^a \sin^2\left(\frac{\pi x}{a}\right) e^{-i\omega t} e^{i\omega t}\,dx$$

$$= \int_0^a \frac{\left(1 - \cos\frac{2\pi x}{a}\right)}{2}\,dx$$

$$= \frac{1}{2}\left[x - \sin\left(\frac{2\pi x}{a}\right)\cdot\frac{a}{2\pi}\right]_0^a$$

$$= \frac{1}{2}\Big[$$

We can not use this formula. By this we get prob. in terms of a & we know prob. is

Date :
Page No. :

We can calculate the prob. only when the fucn is normalised.

$$\text{Probability} = \frac{\int_{x_1}^{x_2} |\psi|^2 dx}{\int_{\text{all space}} |\psi|^2 dx}$$

so $$\text{Prob.} = \frac{\int_{a/4}^{3a/4} \sin^2\left(\frac{\pi x}{a}\right) dx}{\int_0^a \sin^2\left(\frac{\pi x}{a}\right) dx} \qquad 0 \le x \le$$

$$= \frac{\cancel{\frac{1}{2}} \int_{a/4}^{3a/4} \left[1 - \cos\left(\frac{2\pi x}{a}\right)\right] dx}{\cancel{\frac{1}{2}} \int_0^a \left[1 - \cos\frac{2\pi x}{a}\right] dx}$$

$$= \frac{\left[x - \sin\frac{2\pi x}{a} \cdot \left(\frac{a}{2\pi}\right)\right]_{a/4}^{3a/4}}{\left[x - \sin\frac{2\pi x}{a} \cdot \left(\frac{a}{2\pi}\right)\right]_0^a}$$

$$\cancel{= \frac{\left(\frac{3a}{4} - 0\right) - \left(\frac{a}{4} + 0\right) - \left(\frac{a}{4} -\right.}{\left(a - \right.}}$$

$$= \frac{\frac{3a}{4} - \frac{a}{2\pi}\sin\left(\frac{2\pi}{a} \times \frac{3a}{4}\right) - \frac{a}{4} + \frac{a}{2\pi}\sin\left(\frac{2\pi}{a} \times \frac{a}{4}\right)}{a - \frac{a}{2\pi}\sin\frac{2\pi}{a} \cdot a - 0 + 0}$$

$$= \frac{\frac{a}{2} - \frac{a}{2\pi}\sin\frac{3\pi}{2} + \frac{a}{2\pi}\sin\frac{\pi}{2}}{a - \frac{a}{2}\sin 2\pi} = \frac{\frac{a}{2} - \frac{a}{2\pi}\left(\frac{1}{2}\right.}{a - 0}$$

$$= \frac{\frac{a}{2} + \frac{a}{2\pi}\frac{1}{2} + \frac{a}{2\pi}1}{a} = \frac{\frac{a}{2} + \frac{a}{2\pi}\left(\frac{1}{2}\right.}{a}$$

$$= \frac{1}{2} + \frac{1}{2\pi}\frac{1}{2} = \frac{\pi + 2}{2\pi}$$

2

= 0.82 A.

Q.31 The wave fucⁿ of a particle is given as

$$\psi(x) = \frac{1}{\sqrt{a}} e^{-\frac{|x|}{a}}$$

(i) find the prob. of finding the particle $-a < x < +a$.

(ii) find the value of b so that the prob. of finding the particle in the range $-b < x < +b$ is 0.5

(i)

$$\text{Prob.} = \frac{\int_{x_1}^{x_2} |\psi|^2 dx}{\int_{\text{all space}} |\psi|^2 dx} = \frac{\int_{-a}^{a} \frac{1}{a} e^{-\frac{2|x|}{a}} dx}{\int \psi^* \psi \, dx}$$

This fucⁿ is Normalised so $\int \psi^* \psi \, dx = 1$

$$\text{So Prob.} = \int_{-a}^{a} \psi^* \psi \, dx = \int_{-a}^{0} \frac{1}{a} e^{\frac{2x}{a}} dx + \int_{0}^{a} \frac{1}{a} e^{-\frac{2x}{a}} dx$$

$$= \frac{1}{a}\left[\frac{e^{\frac{2x}{a}}}{\frac{2}{a}}\right]_{-a}^{0} - \frac{1}{a}\left[\frac{e^{-\frac{2x}{a}}}{\frac{2}{a}}\right]_{0}^{a}$$

$$= \frac{1}{2}(1 - e^{-2}) - \frac{1}{2}(e^{-2} - 1)$$

$$= \frac{1}{2}(2 - 2e^{-2}) = 1 - e^{-2} = 1 - \frac{1}{e^2}$$

$$= \frac{e^2 - 1}{e^2} \quad \text{Ans}$$

~~OR~~

Note:- $\int_{-a}^{a} f(x) dx$ if fucⁿ is even $f(-x) = f(x)$

" " odd then 0

(1) Only if limit is equal to & opposite then break the limits into 2 parts.

"But fucⁿ $e^{-|x|}$ this is neither even nor odd. for -ve region $|x|$ will replace $-x$."

If fucⁿ is Even then $\int_{-a}^{a} f(x) dx = 2\int_{0}^{a} f(x) dx$

" " odd " $\int^{a} f(x) dx = 0$

Date :
Page No. :

(b) $\int_{-b}^{b} \psi^* \psi \, dx = 0.5$

$$\int_{-b}^{b} \frac{1}{a} e^{-\frac{2|x|}{a}} dx = \frac{1}{2}$$

$$\frac{1}{a} \times 2 \times \int_{0}^{b} e^{-\frac{2x}{a}} dx = \frac{1}{2}$$

$$\frac{2}{a} \left[\frac{e^{-\frac{2x}{a}}}{-\frac{2}{a}} \right]_0^b = \frac{1}{2}$$

$$1 - e^{-2b/a} = \frac{1}{2} \Rightarrow 1 - \frac{1}{2} = e^{-\frac{2b}{a}}$$

$$\frac{1}{2} = e^{-2b/a} \Rightarrow e^{2b/a} = 2$$

$$\frac{2b}{a} = \ln 2$$

$$\Rightarrow \boxed{b = \frac{a}{2} \ln 2}$$

Q.4 :- The normalised state of a free particle is ref a wave func $\psi(x,0) = N e^{-\frac{x^2}{2a^2} + i k_0 x}$. Find normalisation constant.

There are 2 type of problems for normalisation c

→ Here N is multiplied in ψ, So we find N by $\int \psi^* \psi \, dx = 1$ that means func is norma

→ If it is N is not given in ψ then we have Ist normalise the func & then find its v

$$\int_{-\infty}^{\infty} \psi^* \psi \, dx = 1$$

$$\int_{-\infty}^{\infty} N^* e^{-\frac{x^2}{2a^2} - i k_0 x} N e^{-\frac{x^2}{2a^2} + i k_0 x} dx = 1$$

$$|N|^2 \int_{-\infty}^{\infty} e^{-\frac{x^2}{a^2}} dx = 1$$

Physics Physics

This func^n is Even.

$$\therefore |N|^2 \int_{-\infty}^{\infty} e^{-\frac{x^2}{a^2}} dx = 1$$

Use $\int_0^{\infty} e^{-x} x^{n-1} dx = \sqrt{n}$, $n > -1$

$$\int_0^{\infty} e^{-\alpha x} x^n dx = \frac{\sqrt{n+1}}{\alpha^{n+1}} , \quad n > -1$$

$$\int_0^{\infty} e^{-\alpha x^2} x^n dx = \frac{\sqrt{\frac{n+1}{2}}}{2\alpha^{\frac{(n+1)}{2}}}$$

$$\therefore 2|N|^2 \int_0^{\infty} e^{-\frac{x^2}{a^2}} x^0 dx = 1$$

$$\therefore 2|N|^2 \frac{\sqrt{\frac{1}{2}}}{2\left(\frac{1}{a^2}\right)^{\frac{1}{2}}} = 1 \Rightarrow N^2 \frac{\sqrt{\pi}}{\frac{1}{a}} = 1$$

$$\Rightarrow N^2 = \frac{1}{a\sqrt{\pi}}$$

$$N = \frac{1}{\sqrt{\pi^{1/2} a}} \Rightarrow N = \frac{1}{\pi^{1/4} a^{1/2}} \quad \text{Ans} \quad \left(\sqrt{1/2} = \sqrt{\pi}\right)$$

Q.5:- The wave func^n of a particle at a certain instant is given by $\psi(x) = A \exp\left[-\frac{x^2}{a^2} + ikx\right]$. If P_1 & P_2 denotes the probabilities of finding the particle in the range a to $a+da$ & $2a$ to $2a+da$ respectively. Find the ratio $\frac{P_1}{P_2}$.

$$\psi(x) = A \exp\left[\frac{-x^2}{a^2} + ikx\right]$$

Here interval is small so prob. density will be multiplied by the interval.

$$\frac{P_1}{P_2} = \frac{|\psi|^2_{x=a}\, da/\cancel{A}}{|\psi|^2_{x=2a}\, da/\cancel{A}}$$

We dont need to calculate Normalization const. bcoz Here is ratio = ? Both wave func^n will have sam normalisation const.

Date :
Page No. :

$$\frac{P_1}{P_2} = \frac{\cancel{|A|^2} \exp\left(\frac{-2x^2}{a^2}\right)_{x=a}}{\cancel{|A|^2} \exp\left(\frac{-2x^2}{a^2}\right)_{x=2c}}$$

$$= \frac{\exp(-2)}{\exp(-8)} = \exp(6)$$

$$\boxed{\frac{P_1}{P_2} = e^6.}$$

This is the ratio of 2 prob. so it may be g
than 1. {otherwise Prob. can never be >1. $P_{max}=1$

* In Q. M., We calculate probabilistic results.
P_1, P_2, P_3, P_4 are the prob of finding the particle
3, 4 then average prob. of finding the all the p
is $P = \frac{P_1+P_2+P_3+P_4}{4}$

Expectation value :- for a dynamical variable
expectation value is

$$\langle \hat{A} \rangle = \frac{\int_{\text{all space}} \psi^* \hat{A} \psi \, dx}{\int_{\text{all space}} \psi^* \psi \, dx}$$

if wave fucⁿ is Nc
then lower part = 1.
if don't know t
~~fucⁿ is Normaliz~~
not then use comp
form

$\hat{x} = x.$
$\hat{p} = -i\hbar\nabla$
$\hat{K.E.} = -\frac{\hbar^2 \nabla^2}{2m}$
$\hat{V} = -\frac{i\hbar}{m}\nabla$
$\hat{V} = V(r)$

if wave fucⁿ is a fucⁿ
of position then we s
multiply by its operat
value.

Q.6. A particle of mass m is confined in the 1 Dim
box extending from $-2L$ to $+2L$. The wave f
of the particle in this state is

$$\psi(x) = \psi_0 \cos\left(\frac{\pi x}{4L}\right)$$

where ψ_0 is constant.

(i) The normalization factor of this wave fun^n is

(a) $A\sqrt{\frac{2}{L}}$ (B) $\sqrt{\frac{1}{4L}}$ (C) $\sqrt{\frac{1}{2L}}$ (D) $\sqrt{\frac{1}{L}}$

(ii) The expectation value of P^2 in this state is

(a) 0 (B) $\frac{\hbar^2\pi^2}{32L^2}$ (C) $\frac{\hbar^2\pi^2}{16L^2}$ (D) $\frac{\hbar^2\pi^2}{8L^2}$

(i) The Normalization factor :-

$$\int_{-\infty}^{+\infty} \psi^* \psi \, dx = 1$$

Particle is confined in a box i.e. in region from $-2L$ to $+2L$.
So wave fun^n outside the box will be zero.

$$So \int_{-2L}^{+2L} |\psi_0|^2 \cos^2\left(\frac{\pi x}{4L}\right) dx = 1$$

{For particle in a box, by Normalisation factor we can solve this Que. easily by $N = \sqrt{\frac{2}{\text{width}}}$

$$|\psi_0|^2 \int_{-2L}^{2L} \frac{1}{2}\left(1 + \cos\frac{2\pi x}{4L}\right) dx = 1$$

$$\frac{1}{2}|\psi_0|^2 \left[x + \sin\left(\frac{2\pi x}{4L}\right) \times \frac{4L}{2\pi}\right]_{-2L}^{+2L} = 1$$

$$\frac{1}{2}|\psi_0|^2\left[2L - \frac{4L}{2\pi}\sin\left(\frac{2\pi \cdot 2L}{4L}\right) + 2L + \frac{4L}{2\pi}\sin\left(-\frac{2\pi}{4L}\cdot 2L\right)\right] =$$

$$\frac{1}{2}\left[4L - 0\right]|\psi_0|^2 = 1$$

$$|\psi_0|^2 = \frac{2}{4L} = \frac{1}{2L}$$

$$|\psi_0| = \sqrt{\frac{1}{2L}} \quad \text{Ans}$$

(C) → correct

Date :
Page No. :

Expectation value

$$\langle p^2\rangle = \int_{-2L}^{+2L} \psi^*(x)\left(-\hbar^2 \frac{d^2}{dx^2}\right) \psi(x)\, dx.$$

By another method, for partical in a box,

$$E_n = \frac{n^2\pi^2\hbar^2}{2m(\text{width})^2}$$

$$E = \frac{p^2}{2m} \quad \text{or} \quad \langle E\rangle = \frac{\langle p^2\rangle}{2m}$$

So we can calcultate the averge valu mom.

$$\langle p^2\rangle = \int_{-2L}^{2L} \psi_0^* \cos\left(\frac{\pi x}{4L}\right)(-\hbar^2)\frac{d^2}{dx^2}\left[\psi_0 \cos\left(\frac{\pi x}{4L}\right)\right] dx$$

$$\langle p^2\rangle = -\hbar^2 \psi_0^*\psi_0 \int_{-2L}^{2L} \cos\left(\frac{\pi x}{4L}\right) \frac{d}{dx}\left[-\sin\frac{\pi x}{4L}\right]\times\left[\frac{\pi}{4L}\right] dx$$

$$= \hbar^2 \frac{1}{2L}\int_{-2L}^{2L} \cos\frac{\pi x}{4L} \cos\frac{\pi x}{4L} \times \left(\frac{\pi}{4L}\right)^2 dx$$

$$= \frac{\hbar^2}{2L}\left(\frac{\pi}{4L}\right)^2 \cancel{2}\int_0^{2L} \frac{1+\cos\frac{2\pi x}{4L}}{\cancel{2}}\, dx = \frac{\hbar^2}{2L}\left(\frac{\pi}{4L}\right)^2\left[x + \ldots\right.$$

$$= \frac{\hbar^2}{2L}\left(\frac{\pi}{4L}\right)^2\left[2L + \frac{2L}{\pi}\sin\frac{\pi}{2L}\times 2L\right] = \frac{\hbar^2}{\cancel{2}}\left(\frac{\pi}{4L}\right)^2 \cancel{2L}$$

$$= \frac{\hbar^2\pi^2}{16L^2} \quad \text{Ans}$$

✓(C) - correct

Q.7 :- Calcultate the expectation value of P & P^2 for the wave fu^n

$$\psi(x) = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{\pi x}{L}\right), \quad 0 < x < L$$

$$= 0 \qquad \text{otherwise.}$$

$$\langle p\rangle = \int_0^L \psi^*(x)\left(-i\hbar \frac{d}{dx}\right)\psi(x)$$

$$= \int_0^L \sqrt{\frac{2}{L}} \sin\frac{\pi x}{L}\left(-i\hbar\frac{d}{dx}\right)\left(\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}\right) dx$$

$$= \frac{2}{L}\int_0^L \left[\sin\frac{\pi x}{L}(-i\hbar)\cos\left(\frac{\pi x}{L}\right)\times\frac{\pi}{L}\right] dx$$

$$= \frac{\cancel{2}}{L}(-i\hbar)\frac{\pi}{L}\int_0^L \frac{1}{\cancel{2}} \sin\frac{2\pi x}{L}\,dx$$

$$= \frac{\pi}{L^2}(-i\hbar)\left[\cos\frac{2\pi x}{L}\right]_0^L \left(\frac{L}{2\pi}\right)$$

$$= \frac{-i\hbar}{2L}\left[\cos 2\pi - \cos 0\right] = \frac{-i\hbar}{2L}(1-1) = 0$$

$$\langle p^2 \rangle = \int_0^L \psi^* \left(-\hbar^2 \frac{d^2}{dx^2}\right)\psi \, dx$$

$$= -\hbar^2 \int_0^L \frac{2}{L} \sin\frac{\pi x}{L} \frac{d}{dx}\left[-\cos\frac{\pi x}{L}\right] \times \left(\frac{\pi}{L}\right) dx$$

$$= \hbar^2 \frac{2}{L}\int_0^L \sin\frac{\pi x}{L} \sin\frac{\pi x}{L} \times \left(\frac{\pi}{L}\right)^2 dx$$

$$= \frac{2\hbar^2}{L}\left(\frac{\pi}{L}\right)^2 \int_0^L \sin^2\frac{\pi x}{L}\,dx = \frac{2\pi^2\hbar^2}{L^3}\int_0^L \frac{1-\cos 2\frac{\pi x}{L}}{2}$$

$$= \frac{\cancel{2}\pi^2\hbar^2}{L^3 \cancel{2}}\left[x - \frac{\sin\frac{2\pi x}{L}}{\frac{2\pi}{L}}\right]_0^L = \frac{\pi^2\hbar^2}{L^3}\left[L - \frac{\sin 2\pi}{2\pi/L} - 0 + 0\right]$$

$$= \frac{\pi^2\hbar^2}{L^2}$$

Q.8:- Find the expectation value of position & mom. of a particle whose wave fucn is

$$\psi(x) = e^{-\frac{x^2}{a^2} + ikx} \quad \text{in all space.}$$

Calculate $\langle p \rangle$ & $\langle x \rangle$.

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi^* x \psi$$

$$= \int_{-\infty}^{\infty} e^{-\frac{x^2}{a^2} - ikx} x \, e^{-\frac{x^2}{a^2} + ikx} dx$$

$$= \int_{-\infty}^{\infty} x e^{-2\frac{x^2}{a^2}} dx = \int_{-\infty}^{\infty} \underbrace{x}_{odd} \underbrace{e^{-2\frac{x^2}{a^2}}}_{even} dx$$

$$= \frac{\Gamma 1}{2\left(\frac{2}{a^2}\right)^1} = \frac{\Gamma 1}{\frac{4}{a^2}} = 0 \quad (\Gamma 1 = 0)$$

bcoz this wave fucn is odd so integral will be 0

$$\langle x \rangle = 0$$

Date :
Page No. :

$$\langle p\rangle = \frac{\int_{-\infty}^{\infty} e^{-\frac{x^2}{a^2}-ikx}\left(-i\hbar\frac{d}{dx}\right)e^{-\frac{x^2}{a^2}+ikx}\,dx}{\int_{-\infty}^{\infty} e^{-\frac{x^2}{a^2}-ikx}\; e^{-\frac{x^2}{a^2}+ikx}\,dx}$$

$$\frac{-i\hbar\int_{-\infty}^{\infty} e^{-\frac{x^2}{a^2}-ikx}\; e^{-\frac{x^2}{a^2}+ikx}\left(-\frac{2x}{a^2}+ik\right)dx}{2\int_{0}^{\infty} e^{-\frac{2x^2}{a^2}}\,x^0\,dx}$$

$$= \frac{-i\hbar\int_{-\infty}^{\infty} e^{-\frac{2x^2}{a^2}}\left(\frac{-2x}{a^2}+ik\right)dx}{2\int_0^{\infty} e^{-2x^2/a^2}\,dx}$$

odd

$$= \frac{-i\hbar\,\cancel{2}\int_0^{\infty} e^{-2x^2/a^2}\,ik\,dx + 0}{\cancel{2}\int_0^{\infty} e^{-2x^2/a^2}\,dx} = \frac{\hbar k\int_0^{\infty} e^{-2x^2/a^2}}{\int_0^{\infty} e^{-2x^2/a^2}\,dx}$$

$$= \hbar k$$

$$\left\{-\frac{2}{a^2}\int_0^{\infty}(i\hbar)\,x\,e^{-2x^2/a^2}\,dx = 0\right.$$

(odd fuc^n)

Note :- for odd integral when we put $-x$ in place of x then value of fuc^n will be change. The value of this type of integral $= 0$.

But for even integral $f(-x)=f(x)$ then value of this type of integral is Non-zero.

Q.9 :- A free particle of mass m moves along the x-direction at $t=0$ then normalized wave fu^n of the particle is given by
(GATE)

(This wave fuc^n is not normalised)
$$\psi(x,0) = \frac{1}{(2\pi\alpha)^{1/4}}\exp\left[\frac{-x^2}{4\alpha^2}+ix\right]$$

where α is real const.

(i) The expectation value of mom. in this state is (a) $\hbar\alpha$ ✓(b) $\hbar\sqrt{\alpha}$ (c) α (d) $\frac{\hbar}{\sqrt{\alpha}}$

(2) The expectation value of the particle energy is

(a) $\frac{\hbar^2}{2m(2\alpha^{3/2})}$ (b) $\frac{\hbar^2\alpha^2}{2m}$ ✓(c) $\frac{\hbar^2}{2m}\frac{(4\alpha^2+1)}{4\alpha^{3/2}}$ (d) $\frac{\hbar^2}{8m\alpha^3}$

(1) $$\langle p\rangle = \int_{-\infty}^{\infty}\frac{1}{(2\pi\alpha)^{1/4}}\exp\left[\frac{-x^2}{4\alpha^2}-ix\right]\left(-i\hbar\frac{d}{dx}\right)\left(\frac{1}{2\pi\alpha}\right)^{1/4}\exp\left[\frac{-x^2}{4\alpha^2}+ix\right]dx$$

{~~Wave fcn is Normalized so~~ $\langle p\rangle = \int\psi^*\hat{p}\,\psi\,dx$}

$$= \frac{-i\hbar}{(2\pi\alpha)^{1/2}}\int_{-\infty}^{\infty}\exp\left(\frac{-x^2}{4\alpha^2}-ix\right)\exp\left(\frac{-x^2}{4\alpha^2}+ix\right)\left(\frac{-2x}{4\alpha^2}+i\right)dx$$

$$= \frac{-i\hbar}{(2\pi\alpha)^{1/2}}\int_{-\infty}^{\infty}e^{-\frac{x^2}{2\alpha^2}}\left(\frac{-2x}{4\alpha^2}+i\right)dx$$

$$= \frac{-i\hbar}{(2\pi\alpha)^{1/2}}\,2\int_0^{\infty}e^{-\frac{x^2}{2\alpha^2}}x^0\,i\,dx + 0$$

$$= \frac{2\hbar}{(2\pi\alpha)^{1/2}}\cdot\frac{\Gamma\frac{1}{2}}{2\left(\frac{1}{2\alpha^2}\right)^{1/2}} = \frac{2\hbar}{(2\pi\alpha)^{1/2}}\frac{\sqrt{\pi}}{\left(\frac{2}{\alpha^2}\right)^{1/2}}$$

$$= \hbar\frac{2\alpha}{2\alpha^{1/2}} = \hbar\sqrt{\alpha}$$ Ans (b) ✓.

$\langle p\rangle = \frac{\int\psi^*\hat{p}\,\psi\,dx}{\int\psi^*\psi\,dx}$ will give same result.

(2) $$\langle E\rangle = \int_{-\infty}^{\infty}\psi^*\hat{E}\,\psi\,dx$$

$$= \int_{-\infty}^{\infty}\frac{1}{(2\pi\alpha)^{1/4}}\exp\left[\frac{-x^2}{4\alpha^2}-ix\right]\left(\frac{-\hbar^2}{2m}\nabla^2\right)\frac{1}{(2\pi\alpha)^{1/4}}\exp\left[\frac{-x^2}{4\alpha^2}+ix\right]dx$$

$$= \frac{-\hbar^2}{2m(2\pi\alpha)^{1/2}}\int_{-\infty}^{\infty}\exp\left(\frac{-x^2}{4\alpha^2}-ix\right)\exp\left(\frac{-x^2}{4\alpha^2}+ix\right)\left(\frac{-2}{4\alpha^2}\right)dx$$

Date :
Page No. :

$$= \frac{\cancel{2}\hbar^2}{\cancel{2}m(2\pi\alpha)^{1/2}4\alpha^2}\, 2\int_{0}^{\infty} e^{-\frac{x^2}{2\alpha^2}} x^0\, dx$$

$$= \frac{\cancel{2}\,\hbar^2\, 2}{(2\pi\alpha)^{1/2}4\alpha^2 m} \cdot \frac{\Gamma\frac{1}{2}}{2\left(\frac{1}{2\alpha^2}\right)^{1/2}}$$

$$= \frac{\hbar^2}{2\alpha^2 m(2\pi\alpha)^{1/2}}\, \frac{\sqrt{\pi}(2\alpha^2)^{1/2}}{2} = \frac{\hbar^2}{4\alpha m\alpha^{1/2}}$$

$$= \frac{\hbar^2}{2m(2\alpha^{3/2})} \quad \text{Ans.} \qquad (a)$$

$$\psi(x,0) = \frac{1}{(2\pi\alpha)^{1/4}} \exp\left[-\frac{x^2}{4\alpha^2} + ix\right]$$

$$\frac{d^2\psi}{dx^2} = \frac{d}{dx}\left[\frac{1}{(2\pi\alpha)^{1/4}} \exp\left(-\frac{x^2}{4\alpha^2}+ix\right)\left(\frac{-2x}{4\alpha^2}+i\right)\right.$$

$$= \frac{1}{(2\pi\alpha)^{1/4}}\left[\exp\left(-\frac{x^2}{4\alpha^2}+ix\right)\left(\frac{-2}{4\alpha^2}\right) + \exp\left(-\frac{x^2}{4\alpha^2}+ix\right)\right.($$

$$\langle E\rangle \cancel{\psi^* \frac{d^2\psi}{dx^2}} = \int_{-\infty}^{\infty}\psi^*\left(-\frac{\hbar^2}{2m}\nabla^2\right)\psi\, dx = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\psi^* \frac{d^2\psi}{dx^2}$$

$$= \frac{-\hbar^2}{2m}\int_{-\infty}^{\infty}\frac{1}{(2\pi\alpha)^{1/2}}\left[\frac{-2}{4\alpha^2}e^{-\frac{2x^2}{4\alpha^2}} + \left(\frac{-2x}{4\alpha^2}+i\right)e^{-2x^2/4\alpha^2}\right]dx$$

$$= \frac{-\hbar^2}{2m(2\pi\alpha)^{1/2}}\int$$

$$-i\hbar\frac{\partial}{\partial x} \qquad = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

- If we know the wave fucⁿ then we can calculate the expectation value of any dynamical variable.
- If we know the wave fucⁿ we can calculate the uncertainity product for that fucⁿ.

Deviation :- $\Delta x = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$

Root mean square deviation or uncertainity deviation.

$$\Delta x = \sqrt{\langle (x^2 + \langle x \rangle^2 - 2x\langle x \rangle) \rangle}$$

Here $\langle x \rangle$ i.e. x average is constant i.e. $\langle \langle x \rangle^2 \rangle = \langle x \rangle^2$

$$= \sqrt{\langle x^2 \rangle + \langle x \rangle^2 - 2\langle x \rangle \langle x \rangle}$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

So this is the uncertainity product. We can also calculate the Varience,

$$\boxed{\text{Varience} = \langle x^2 \rangle - \langle x \rangle^2}$$

Note :- Deviation means $x - \langle x \rangle$
& its standard form is $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

To calculate the uncertainity product for mom.,

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

If we know the state then we $\Delta x \Delta p \simeq \hbar$ for exact state. but in general if we dont know the state then $\Delta x \Delta p \geq \frac{\hbar}{2}$. (there will be inequality)

* $\Psi(x) = A\,\delta(x-x_0)$

dirac-delta func^n,

$\delta(x-x_0) = 0 \quad , \quad x \neq x_0$

$\quad\quad\quad = \infty \quad\quad x = x_0$

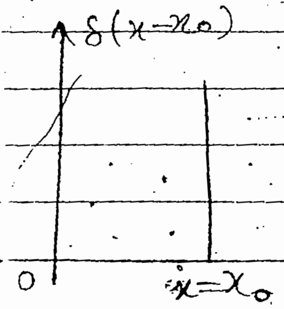
If we check, then this is not a physically a
~~func^n so we can not sep^n this func as particle.~~

The prob. of finding the particle at $x = x_0$

$\Delta x = 0$

$\Rightarrow \quad \Delta p = \infty$

If we know the wave func^n in one state then can convert it into wave func^n for another st



There is no spread in wave func^n & there is no spread, there is only single peak at $x=x_0$
Hence uncertainity in position is zero.

$\Delta x = 0$

& $\Delta x \Delta p \simeq \hbar \Rightarrow \Delta p = \infty$

$\Rightarrow$ When dirac delta func^n is in mom., then

$\phi(p) = A\,\delta(p-p_0)$

then $\quad \Delta x = \infty$

$\quad\quad\quad \Delta p = 0$

bcoz there is no spreading at $p = p_0$.

Schroedinger Equation :-

1) Most general Sch^r eq^n
2) Time independent Sch^r eq^n
3) Time dependent Sch^r eq^n

There is only one Sch^n eq^n which is most general Sch^r eq^n. & we can treat it as time dependent or time ~~eq~~ independent. $\boxed{-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi}$

⇒ If V in independent of time $V = V(r)$
then we use Sch^r eq^n

$$H\psi(r,t) = E\psi(r,t)$$

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\psi(r,t) = i\hbar\frac{\partial}{\partial t}\psi(r,t)$$

$$\psi(r,t) = \phi(r)\, f(t)$$

suppose wave func^n is the product of 2 func^n, one is func^n of position & one of time.

$$\Rightarrow f(t)\left[\frac{-\hbar^2}{2m}\nabla^2\phi(r) + V(r)\phi(r)\right] = i\hbar\,\phi(r)\frac{\partial f}{\partial t}$$

$$\frac{f(t)\left[\frac{-\hbar^2}{2m}\nabla^2 + V(r)\right]\phi(r)}{\phi(r)\,f(t)} = \frac{i\hbar\,\phi(r)\frac{\partial f(t)}{\partial t}}{\phi(r)\,f(t)}$$

(divide by $\phi(r)\,f(t)$ on both side)

$$\Rightarrow \frac{1}{\phi(r)}\left[\frac{-\hbar^2}{2m}\nabla^2 + V(r)\right]\phi(r) = \frac{i\hbar\frac{\partial f(t)}{\partial t}}{f(t)} \quad \text{——(1)}$$

L.H.S. depends on position & R.H.S. depend on time.

* If pot^n is time dependent $V(r,t)$.
then we can not use variable seperable method. i.e. we can not write
$$\psi(x,t) = \phi(r)\,f(t)$$

Now using ~~m~~ seperation of variable method in (1)

Date :
Page No. :

$$i\hbar \frac{1}{f(t)} \frac{\partial f}{\partial t} = E$$

$$\frac{\partial f}{f} = \frac{-iE}{\hbar}\partial t$$

On integrating $\ln f = \frac{-iE}{\hbar} t + \ln A$

$$\ln \frac{f}{A} = \frac{-iEt}{\hbar}$$

taking antilog, $\boxed{f(t) = A e^{-iEt/\hbar}}$ — (2

$\psi(r,t) = \phi(\cdot)f(t) = A\phi(r)e^{...}$

Substitute this value in most general ~~&~~ Sch^r

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\psi(r,t) = i\hbar \frac{\partial}{\partial t}\psi(r,t)$$

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]A\phi(r)e^{-iEt/\hbar} = i\hbar \frac{\partial}{\partial t}\left[A\phi(r)e\right.$$

We have.- $\psi(r,t) = A\phi(r)e^{-iEt/\hbar}$

$$i\hbar \frac{\partial \psi(r,t)}{\partial t} = i\hbar \times A\phi(r)\left(\frac{-iE}{\hbar}\right)e^{-iEt/\hbar}$$

$$= E A \phi(r) e^{-iEt/\hbar}$$

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right] A\phi(r)\cancel{e^{-iEt/\hbar}} = EA\phi\, \cancel{e^{-iEt/}}$$

~~time dependent~~ part cancel out from both side & time dependancy will ~~cancel out~~ elim

$$\boxed{\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\psi(r,t) = E\psi(r,t)}$$

This is Time-Independent Sch^r eq^n.

OR $\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\phi(r) = E\phi(r)$

If $V = V(r,t)$ then

$$~~\psi(r,t) \neq \phi(r) f(t)~~$$

$$f(t) \neq Ae^{-iEt/\hbar}$$

then we get

$$\boxed{\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r,t)\right]\psi(r,t) = i\hbar\frac{\partial}{\partial t}\psi(r,t)}$$

from this eqn, we can not eliminat time par
This is Time Dependent schr eqn.

Hence, weather the schr eqn is Time dependent or time independent, it depends on potential.

$V=V(r)$ If potn is time independent then schr eqn is time independent.

$V=V(r,t)$ If potn is time dependent then schr eqn is time dependent.

Stationary State - The state depanding on time but probability is independent or time i.e. potn is independent on time, is called stationary state i.e.
"The state in which prob. density or probability of finding the particle is independent of time, is called Stationary state."

$$\psi(r,t) = A\,\phi(r)\,e^{-iEt/\hbar}$$

$$\psi^*\psi = A^*A\,\phi^*(r)\,\phi(r)$$

$$|\psi|^2 = |A|^2|\phi(r)|^2$$

i.e. Prob. density is defined by this state i.e. independent of time. (but state show the dependance

Date :
Page No. :

Also $\int \psi^* \psi \, d\tau \Rightarrow$ Time independent

for Stationary state $\frac{\partial A}{\partial t} = 0$ independent of ti

if at $t=0$, prob $= 1/2$ & energy $= 1eV$ ~~they never~~ changes, always remain like t ~~not depending on time.~~

for more than 1 state, we calculate the av energy. In different state, system's properties change but its average will be same.

Application of Schroedinger eqn :- There are

(1) If we know the wave funcn, we can calcul potn.

i.e. $\psi(x,t) =$ given then $V(x,t) = ?$

(2) If potn is given then we can calculate th wave funcn

$V(x) =$ given , $\psi(x,t) = ?$

* Only In time dependent perturbation theory potn is time dependent otherwise in all other cases potn is time independent & except this case, always use time indepe Schr eqn.

(1) To calculate potn in Ist type of problem, use energy value eqn $H\psi = E\psi$

When $E =$ const. , $E \rightarrow$ energy eigen value

means E is independent on that variable on which ψ is depending

i.e. if ψ depend on position then E should not dep on po

Prob 2. The wave func^n of a spin-less particle of mass m in 1 Dim potential is $\psi(x) = A\exp(-\alpha^2 x^2)$ corresponding to an eigen value $E_0 = \frac{\hbar^2\alpha^2}{m}$. The pot^n $V(x)$ is

(a) $2E_0(1-\alpha^2 x^2)$ (b) $2E_0(1+\alpha^2 x^2)$

(c) $2E_0\alpha^2 x^2$ (d) $2E_0(1+2\alpha^2 x^2)$

$\psi(x) \to$ given

$V(x) \to$?

Use time independent Sch^r eq^n

$$H\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi \quad \text{——(1)}$$

$$\psi = A\exp(-\alpha^2 x^2)$$

Wave func^n depending on position & E will be the eigen value if it is independent of position i.e. R.H.S. part should be " " "

i.e. after operating, ~~the obtained L.H.S. part~~ we have to make it independent of position.

Now $\frac{d^2\psi}{dx^2} = \frac{d}{dx}\left(\frac{d\psi}{dx}\right) = \frac{d}{dx} A\exp(-\alpha^2x^2)(-2\alpha^2 x)$

$= A\left[\exp(-\alpha^2 x^2)(-2\alpha^2 x)^2 + \exp(-\alpha^2x^2)(-2\alpha^2\right.$

$= 4A\alpha^4 x^2\exp(-\alpha^2x^2) - 2\alpha^2 A\exp(-\alpha^2$

Eq^n (1) $\Rightarrow$ $-\frac{\hbar^2}{2m}\left[4A\alpha^4x^2 - 2\alpha^2 A\right]\exp(-\alpha^2x^2) + V(x)A\exp(-\alpha^2x^2)$

$= EA\exp(-\alpha^2x^2)$

$\Rightarrow -\frac{\hbar^2}{2m}\left[4\alpha^4x^2 - 2\alpha^2\right] + V(x) = E$

Date :
Page No. :

$H\psi = \underline{E}\,\psi$

This E is energy eigen value Then H mu
position independent as ψ is depending on ψ

Put The position depending part is zero

$$V(x) - \frac{\hbar^2}{2m}4\alpha^4 x^2 = 0$$

$$\Rightarrow V(x) = 4\frac{\hbar^2}{2m}\alpha^4 x^2$$

$$V(x) = 2E_0\alpha^2 x^2 \qquad \{ \text{as } E_0$$

(c) is correct.

Prob: The wave func^n of a particle moving in a 1 D
time independent pot V(x) is
$\psi(x) = \exp(-iax+b)$ where a & b are const
This means that the pot^n V(x) is of the form

~~a) $V(x) \propto x$~~ ~~(b) $V(x) \propto x^2$~~

c) $V(x) \cancel{\propto} 0$ (d) $V(x) \propto e^{-ax}$

(e) $V(x) = V_0$

$$\psi(x) = e^{-iax+b}, \quad V(x) = ?$$

Time Independent Sch^r eq^n

$$H\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\left[\frac{d}{dx}\left\{e^{-iax+b}(-ia)\right\}\right] + V(x)\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\left[e^{-iax+b}(-ia)^2\right] + V(x)\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\left[a^2 e^{-iax+b}\right] + V(x)e^{-iax+b} = E e$$

$$-\frac{\hbar^2}{2m}a^2 + V(x) = E$$

Equate position dependent part equal to zero.

i.e. $\boxed{V(x) = 0}$

(c) is correct.

Prob :- For a particle of mass m in 1 Dim harmonic oscillator potⁿ $V(x) = \frac{1}{2} m\omega^2 x^2$. The first excited energy eigen state is $\psi(x) = x\, e^{-ax^2}$. The value of a is

(A) $\frac{\omega m}{4\hbar}$ (B) $\frac{m\omega}{3\hbar}$ (C) $\frac{m\omega}{2\hbar}$ (d) $\frac{2m\omega}{3\hbar}$

Trick - Wave funcⁿ of harmonic oscillator

$$\psi(x) = e^{-\frac{\alpha^2 x^2}{2}}$$

$$\& \ \alpha^2 = \frac{m\omega}{\hbar}$$

Compare this wave funcⁿ with given $\psi(x) = x e^{-ax^2}$

$$\frac{\alpha^2}{2} = a \Rightarrow \alpha^2 = 2a$$

$$\frac{m\omega}{\hbar} = 2a \Rightarrow \boxed{a = \frac{m\omega}{2\hbar}}$$

Now By Method, $H\psi = E\psi$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

$$\frac{-\hbar^2}{2m}\left[\frac{d}{dx}\left\{x e^{-ax^2}(-2ax) + e^{-ax^2}\right\}\right] + V(x)\psi = E\psi$$

$$\frac{-\hbar^2}{2m}\left[\frac{d}{dx}\left\{-2ax^2 e^{-ax^2}\right\} + e^{-ax^2}\right] + V(x)\psi = E\psi$$

$$\frac{-\hbar^2}{2m}\left[-2ax^2 e^{-ax^2}(-2ax) - 2ae^{-ax^2}(2x) + e^{-ax^2}(-2ax)\right] + V(x)\psi = E\psi$$

Date:
Page No.:

$$\frac{-\hbar^2}{2m}\left[4a^2x^2 \mp 4a - 2a\right] x e^{-ax^2} + V(x)\, x e^{-ax^2}$$

$$\frac{-\hbar^2}{2m}\left[4a^2x^2 \pm 6a\right] + V(x) = E$$

$$-\frac{\hbar^2}{2m}\left[4a^2x^2 - 6a\right] + \frac{1}{2}m\omega^2 x^2 = E$$

Position depending part

$$-\frac{\hbar^2}{2m} 4a^2x^2 + \frac{1}{2} m\omega^2 x^2 = 0$$

$$+\hbar^2 \frac{2a^2}{m} = \frac{1}{2} m\omega^2$$

$$a^2 = \frac{m^2\omega^2}{4\hbar^2}$$

$$\Rightarrow \boxed{a = \frac{m\omega}{2\hbar}}$$

✓ (C) is correct.

Prob: The ground state (apart from normalization) of particle of unit mass moving in a 1 Dim potⁿ V(x) is $\exp\left(-\frac{x^2}{2}\right)\cosh(\sqrt{2}x)$. The potⁿ V(x) in suitable unit so that $\hbar = 1$ is

(a) $\frac{x^2}{2}$ ✓(b) $\frac{x^2}{2} - \sqrt{2}x\tanh(\sqrt{2}x)$

(c) $\frac{x^2}{2} - \sqrt{2}x\tan(\sqrt{2}x)$ (d) $\frac{x^2}{2} - \sqrt{2}x\coth(\sqrt{2}x)$

$$\psi(x) = e^{-x^2/2}\cosh(\sqrt{2}x)$$

Time independent schʳ eqⁿ $H\psi = E\psi$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

$$\frac{-\hbar^2}{2m}\left[\frac{d}{dx}\left\{e^{-x^2/2}(-x)\cosh\sqrt{2}x + e^{-x^2/2}\sqrt{2}\sinh(\sqrt{2}x)\right\}\right]$$

$$\cancel{\frac{-\hbar^2}{2m}}\Big[$$

$$\frac{-\hbar^2}{2m}\Big[-x\cosh\sqrt{2}x \cdot e^{-x^2/2}(-x) + e^{-x^2/2}\big[-\cosh\sqrt{2}x - \sqrt{2}x\sinh\sqrt{2}x\big]$$

$$+\sqrt{2}e^{-x^2/2}\cosh\sqrt{2}x + \sqrt{2}e^{-x^2/2}(-x)\sinh\sqrt{2}x\Big] + V(x)\psi = E\psi$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left[x^2 - 1 - \sqrt{2}x\tanh\sqrt{2}x + \sqrt{2} - \sqrt{2}x\tanh\sqrt{2}x\right]e^{-x^2/2}$$

$$+ V(x)e^{-x^2/2}\cosh\sqrt{2}x = E e^{-x^2/2}\frac{\cosh\sqrt{2}x}{\cosh\sqrt{2}x}$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left[x^2 + (\sqrt{2}-1) - 2\sqrt{2}x\tanh(\sqrt{2}x)\right] + V(x) = E$$

$$-\frac{\hbar^2}{2m}\left[x^2 - 2\sqrt{2}x\tanh(\sqrt{2}x) + (\sqrt{2}-1)\right] + V(x) = E$$

Position dependent part $= 0$

$$-\frac{\hbar^2}{2m}\left[x^2 - 2\sqrt{2}x\tanh\sqrt{2}x\right] + V(x) = 0$$

Given $\hbar = 1$ & $m = 1$ (unit mass)

$$\Rightarrow V(x) = \frac{1}{2}\left[x^2 - 2\sqrt{2}x\tanh(\sqrt{2}x)\right]$$

$$\Rightarrow \boxed{V(x) = \frac{x^2}{2} - \sqrt{2}\,x\tanh(\sqrt{2}x)}$$ (b) → correct.

Probability Current density :-
or
Equation of Continuity in Q.M. :-

In Electrodynamics,

If we know the volume charge density ρ & current charge density J then continuity eqn in electrodynamics

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot J = 0}$$

$$\nabla\cdot J = -\frac{\partial\rho}{\partial t}$$

divergence repn the slope & slop may be +ve or −ve.

Date :

Page No. :

eq^n of continuity (exp^n) shows the charge conser

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot J$$

If $\nabla \cdot J = +ve$ $\frac{\partial \rho}{\partial t} = -ve$ charge density w.r.to t

If $\nabla \cdot J = -ve$ then

$$\frac{\partial \rho}{\partial t} = +ve$$ i.e. charge density w.r.t

In Q. Mech., $\rho \rightarrow p$

$J \rightarrow S$

The probability of finding the particl at a particu point in space $= \psi^*(r,t)\,\psi(r,t)$ is known,

Then the prob of finding the particle in a region space volume V bounded by the surface area A &

$$P = \int_V \psi^*(r,t)\,\psi(r,t)\,d\tau = \int_V p\,d\tau$$

$p \rightarrow exp^n$ the probability density.

Change in prob. w.r.to time,

$$\frac{\partial P}{\partial t} = \int_V \left[\left(\frac{\partial \psi^*}{\partial t}\right)\psi + \psi^*\left(\frac{\partial \psi}{\partial t}\right)\right] d\tau \qquad \text{——(1)}$$

To calculate these terms, we use Sch^n eq^n.

$$H\psi = E\psi$$

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r,t)\right]\psi(r,t) = i\hbar\frac{\partial \psi}{\partial t} \qquad \text{——(A)}$$

In Case of Complex pot^n

$$\frac{\partial p}{\partial t} + \nabla \cdot S \neq 0$$

For Real pot^n $\frac{\partial p}{\partial t} + \nabla \cdot S = 0$ (Prob. will con

Date :
Page No. :

In case of complex potⁿ there will be transmission, reflection as well as absorbance so probability will not be conserved.

$$(A) \Rightarrow \frac{\partial \psi}{\partial t} = \frac{-i}{\hbar}\left[-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi\right]$$

$$\frac{\partial \psi^*}{\partial t} = \frac{i}{\hbar}\left[-\frac{\hbar^2}{2m}\nabla^2\psi + V^*\psi\right]$$

$$\psi^*\frac{\partial \psi}{\partial t} = \psi^*\left(\frac{-i}{\hbar}\right)\left[\frac{-\hbar^2}{2m}\nabla^2\psi + V\psi\right] \quad \text{---(2)}$$

$$\psi\frac{\partial \psi^*}{\partial t} = \psi\left(\frac{i}{\hbar}\right)\left[\frac{-\hbar^2}{2m}\nabla^2\psi + V^*\psi\right] \quad \text{---(3)}$$

eqⁿ (2) $\Rightarrow$ + eqⁿ (3)

$$\psi^*\frac{\partial \psi}{\partial t} + \psi\frac{\partial \psi^*}{\partial t} = -\frac{i}{\hbar}\left[\frac{-\hbar^2}{2m}\psi^*\nabla^2\psi + \psi^* V\psi\right] + \frac{i}{\hbar}\left[\frac{-\hbar^2}{2m}\psi\nabla^2\psi^* + \psi V^*\psi^*\right]$$

$$= \frac{-i\hbar}{2m}\left[\psi\nabla^2\psi^* - \psi^*\nabla^2\psi\right]$$

$$\left\{-\frac{i}{\hbar}\left[\psi^* V\psi - \psi V^*\psi^*\right]\right.$$

$V^* = V$ then $\Rightarrow -\frac{i}{\hbar}\left[V\psi^*\psi - V\psi\psi^*\right] = -\frac{i}{\hbar}V(\psi^*\psi - \psi\psi^*) = 0$

for real potⁿ in this term $= 0$

$$= \frac{-i\hbar}{2m}\left[\psi\nabla^2\psi^* - \psi^*\nabla^2\psi + \nabla\psi^*\nabla\psi - \nabla\psi^*\nabla\psi\right]$$

Now eqⁿ (1) $\Rightarrow$

$$\frac{\partial P}{\partial t} = \int_V \nabla\left[\frac{-i\hbar}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi)\right]d\tau$$

$$\frac{\partial}{\partial t}\int_V p\,d\tau = \int_V \frac{\partial p}{\partial t}d\tau = \int_V \nabla\left[\frac{-i\hbar}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi)\right]d\tau$$

$$\Rightarrow \int\frac{\partial p}{\partial t}d\tau + \int\nabla\left[\frac{-i\hbar}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*)\right]d\tau$$

Date :
Page No. :

By the Analogy of Electrodynamics

~~We have~~ $\cancel{\frac{\partial p}{\partial t}} \cancel{= -\nabla \cdot S}$

$$\boxed{S = \frac{-i\hbar}{2m}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)}$$

So $\boxed{\frac{\partial p}{\partial s} + \nabla \cdot S = 0} \Rightarrow \boxed{\frac{\partial p}{\partial t} = -\nabla \cdot S}$

<u>Note</u>: Prob. of finding the particle charge if partic moving.

If particle moves away then prob. ↓

" " inward " " ↑

Suppose, $\psi = e^{ikx}$

$\psi^* = e^{-ikx}$

$\frac{d\psi}{dx} = ike^{ikx}$, $\frac{d\psi^*}{dx} = -ike^{-ikx}$

$$\psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx} = e^{-ikx}\, ike^{ikx} - e^{ikx}(-ik)e^{-ikx}$$
$$= ik - (-ik)$$
$$= 2ik$$

So $S = \frac{-i\hbar}{2m}\left[\psi^* \nabla\psi - \psi\nabla\psi^*\right] = \frac{-i\hbar}{2m} \times 2ik$

$$= \frac{\hbar k}{m} = \frac{p}{m} =$$

$\boxed{S = Re(\psi^* v \psi)}$ This is always valid.

$\boxed{S = v|\psi|^2}$ is valid only when wave fun^n is complex & there is only single term.

$\psi = c\psi'$

for Real part, Current density $= 0$
$\because$ $\psi^* = \psi$ so $\psi^* \nabla \psi - \psi \nabla \psi^* = 0$

Unit of Current density :-

$$S = v |\psi|^2$$

S → is not accctual current. It is current associated with moving particle i.e. wave & wave may be charged or neutral.
So its unit is not equal to the unit of current i.e. Ampere (C/sec)
Its unit is diffrent for 1D, 2D, 3D --- wave fun.

* Sometimes prob. current density S can be interpret as Particle flux.

S = Particle flux.

To calculate particle flux,

$$\oint E \cdot dS = \int (\nabla \cdot E) d\tau$$ by this we get velocity part.

It means simultaneously can not measure position & mom.
If S is independent of position, only then we can interprete ~~a~~ S as particle flux.
This is inconsistant ~~of~~ with Heisenberg uncertainity principle.

$$S = Re(\psi^* v \psi)$$
$$= v \psi^* \psi$$
$$\boxed{S = v |\psi|^2}$$

To calculate the unit of S we have to ~~f~~ know the unit of $|\psi|^2$ in 1 Dim, 2D, 3D ....

Date :
Page No. :

In 1 Dim, Unit of $|\psi|^2 S$ = meter/sec × $\frac{1}{\text{meter}}$ = sec

2 D, = meter / sec × $\frac{1}{(\text{meter})^2}$ = met

3 D, = meter/sec × $\frac{1}{(\text{meter})^3}$ = meter

Que:- A particle of mass m is represented by the wa funcⁿ $\psi(x) = Ae^{ikx}$ where K is the wave ve A is constant. The magnitude of the probabil current density of the particle is

(a) $|A|^2 \frac{\hbar K}{m}$ (b) $|A|^2 \frac{\hbar K}{2m}$

(c) $|A|^2 \frac{(\hbar K)^2}{m}$ (d) $|A|^2 \frac{(\hbar k)^2}{2m}$

$$\psi(x) = Ae^{ikx}$$

$$\Rightarrow \psi^*(x) = A^* e^{-ikx}$$

$$S = -\frac{i\hbar}{2m}\left[\psi^* \nabla \psi - \psi \nabla \psi^*\right]$$

$$= -\frac{i\hbar}{2m}\left[A^* e^{-ikx} A ik e^{ikx} - Ae^{ikx} A^*(-ik)e^{-ikx}\right]$$

$$= -\frac{i\hbar}{2m}\left[AA^* ik + AA^* ik\right]$$

$$= \frac{\hbar K (2|A|^2)}{2m}$$

$$\boxed{S = |A|^2 \frac{\hbar K}{m}}$$ ✓(a).

Qu:- The wave funcⁿ of particle moving in free space is
$\psi = e^{ikx} + 2e^{-ikx}$

(1) The energy of the particle

(a) $\frac{5\hbar^2k^2}{2m}$ (b) $\frac{3\hbar^2k^2}{2m}$ (c) $\frac{\hbar^2k^2}{2m}$ (d) $\frac{\hbar^2k^2}{m}$

(2) The prob. current density for the real part of the wave fucn is

(a) 1 (b) $\frac{\hbar k}{m}$ (c) $\frac{\hbar k}{2m}$ (d) 0

(2)

$$\psi = e^{ikx} + 2e^{-ikx}$$

$$= \cos kx + i\sin kx + 2\cos kx - 2i\sin kx$$

Real part $= \cos kx + 2\cos kx$

$$= 3\cos kx$$

$\left\{ \begin{array}{l} \psi = 3\cos kx \\ \psi^* = 3\cos kx \end{array} \right.$

$$S = \frac{-i\hbar}{2m}\left[\psi^* \nabla\psi - \psi\nabla\psi^*\right]$$

$$=$$

$$S = 0 \qquad (d)$$

$p = -\frac{i}{\hbar}\hbar\frac{\partial}{\partial x}$ $\qquad E = \frac{p^2}{2m}$

(1)

$$E\psi = \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\psi$$

{ We can do this prob. like this but the correct method is given next

$$E\psi = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\left[e^{ikx} + 2e^{-ikx}\right]$$

$$= -\frac{\hbar^2}{2m}\frac{d}{dx}\left[ike^{ikx} + -2ike^{-ikx}\right]$$

$$= \frac{-\hbar^2}{2m}\left[(ik)^2 e^{ikx} + 2(ik)^2 e^{-ikx}\right]$$

$$E\psi = -\frac{\hbar^2}{2m}(ik)^2\left[e^{ikx} + 2e^{-ikx}\right]$$

$$E\psi = -\frac{\hbar^2}{2m} i^2k^2\, \psi$$

$$\boxed{E = \frac{\hbar^2k^2}{2m}} \quad (c)$$

(2) $\psi = 1e^{ikx} + 2e^{-ikx}$, $\langle E\rangle = \frac{1}{5}\frac{\hbar^2k^2}{2m} + \frac{4}{5}\frac{\hbar^2k^2}{2m}$

$\psi = \frac{e^{ikx}+2e^{-ikx}}{\sqrt5}$, $\psi^* = \frac{e^{-ikx}+2e^{ikx}}{\sqrt5}$

$P = \psi\psi^* = \left(\frac{1}{5}+\frac{4}{5}\right) = 1 \Rightarrow E = \sum_r P_r E_r = \frac{\hbar^2k^2}{2m}$

Date:

Page No.:

* If the eigen value for the wave fun^n $\psi = e^{ikx}$ then $H\psi = E\psi$

$$\langle H\rangle = \langle E\rangle = \frac{\int\psi^* H\psi\,dx}{\int\psi^*\psi\,dx} = \frac{\int\psi^* E\psi\,dx}{\int\psi^*\psi\,dx}$$

$|c|^2$

$P = |c|^2$

→ ~~If there is only single state then~~ expectation value = eigen value

→ But if there is more than one state then then there is linear combination of these 2 wave & energy eigen value will be the average of the 2 wave fun^n. So we can do

either by $\langle E\rangle = \frac{\int\psi^* H\psi\,dx}{\int\psi^*\psi}$ or $\langle E\rangle = \sum_r P_r$

Prob. X

(1) $\psi = \frac{1}{\sqrt2}e^{ikx} + \frac{2}{\sqrt2}e^{-ikx}$

$|\psi\rangle = a_1|\psi_1\rangle + a_2|\psi_2\rangle +$

Prob $\langle E\rangle = P_1E_1 + P_2 E_2$

$\frac{\hbar^2k^2}{2m}\left(\frac{1}{2}+\right.$

$\psi^* = e^{-ikx} + 2e^{ikx}$

$$\langle E\rangle = \frac{\int\psi^* H\psi\,dx}{\int\psi^*\psi\,dx} = \frac{\int\psi^*\left(\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\right)\psi\,dx}{\int\psi^*\psi\,dx}$$

$$= \frac{\int\psi^*\left(-\frac{\hbar^2}{2m}\right)i^2k^2\psi\,dx}{\int\psi^*\psi\,dx} = \frac{\hbar^2k^2}{2m}\frac{\cancel{\int\psi^*\psi\,dx}}{\cancel{\int\psi^*\psi\,dx}}$$

$$= \frac{\hbar^2k^2}{2m}$$

Note: When $\psi^*\psi$ = independent of position then we can not normalise this wave fun^n by $\int\psi^*\psi\,dx =$ This can be normalise by dirac-delta cond^n

i.e. $\int\psi_K^*(x)\,\psi_{K'}(x)\,dx = \delta(K-K')$ $\delta(K'-K) = \infty$

$$E = |C_1|^2 \frac{\hbar^2 k^2}{2m} + |C_2|^2 \frac{\hbar^2 k^2}{2m}$$
$$= \frac{1}{5} \qquad + \left(\frac{2}{\sqrt{5}}\right)^2$$

Date :
Page No. :

for limited space, $\int \psi^* \psi \, dx = 1$

free space, $\int \psi_k^*(x) \, \psi_k(x) \, dx = \delta(k-k')$

$\psi = e^{ikx}$. This wave funcⁿ can not repⁿ the particle only it can repⁿ the scattering wave.

Que :- Calculate the prob. current density for a gaussian wave of packet $\psi(x) = A \exp\left[\frac{-x^2}{a^2} + \frac{i p_0 x}{\hbar}\right]$

$$\psi^*(x) = A \exp\left[\frac{-x^2}{a^2} - \frac{i p_0 x}{\hbar}\right]$$

$$S = \frac{-i\hbar}{2m}\left[\psi^* \nabla \psi - \psi \nabla \psi^*\right]$$

$$S = \frac{-i\hbar}{2m}\left[A \exp\left(\frac{-x^2}{a^2} - \frac{i p_0 x}{\hbar}\right) \cdot A\left(\frac{-2x}{a^2} + \frac{i p_0}{\hbar}\right) \exp\left(\frac{-x^2}{a^2} + \frac{i p_0 x}{\hbar}\right)\right.$$
$$\left. - A \exp\left(\frac{-x^2}{a^2} + \frac{i p_0 x}{\hbar}\right) \cdot A\left(\frac{-2x}{a^2} - \frac{i p_0}{\hbar}\right) \exp\left(\frac{-x^2}{a^2} - \frac{i p_0 x}{\hbar}\right)\right]$$

$$= \frac{-i\hbar}{2m}|A|^2\left[\exp\left(-\frac{2x^2}{a^2}\right) \cdot \left(\frac{-2x}{a^2} + \frac{i p_0}{\hbar}\right) - \exp\left(-\frac{2x^2}{a^2}\right) \cdot \left(\frac{-2x}{a^2} - \frac{i p_0}{\hbar}\right)\right]$$

$$= \frac{-i\hbar}{2m}|A|^2 \exp\left(\frac{-2x^2}{a^2}\right)\left[\cancel{\frac{-2x}{a^2}} + \frac{i p_0}{\hbar} + \cancel{\frac{2x}{a^2}} + \frac{i p_0}{\hbar}\right]$$

$$= \frac{-i\cancel{\hbar}}{\cancel{2}m}|A|^2 \exp\left(\frac{-2x^2}{a^2}\right)\left(\frac{\cancel{2} i p_0}{\cancel{\hbar}}\right)$$

$$S = \frac{p_0}{m}|A|^2 \exp\left(\frac{-2x^2}{a^2}\right) \quad \text{Ans}$$

$p = mv$
$v = \frac{p}{m} = \frac{-i\hbar}{m}\frac{\partial}{\partial x}$ $A \exp(\ldots)$
$= \frac{-i\hbar}{m}\left\{\frac{-2x}{a^2} - \frac{i p_0}{\hbar}\right\}$
$= \frac{-i\hbar i p_0}{m \hbar} = \frac{-p_0}{m}$

II method, Velocity of W. funcⁿ $= \frac{p_0}{m}$

$$S = v|\psi|^2 = \frac{p_0}{m}|A|^2 \exp\left(-\frac{2x^2}{a^2}\right)$$

~~If max~~ $P\psi = \lambda\psi$
Eigen value of P should be independent of posi

Que :- Calculate the prob. current density for a 3 dim free^n $\psi(r) = \frac{A}{r} e^{ikr}$ $\{\psi^* = \frac{A}{r} e^{-ikr}$

$$\hat{v} = \frac{-i\hbar}{m}\nabla \Rightarrow v\psi = \frac{-i\hbar}{m}\frac{\partial}{\partial r}\left[\frac{A}{r}e^{ikr}\right]$$

$$v\psi = \frac{-i\hbar}{m} A\left[\frac{ik}{r}e^{ikr} + e^{ikr}\left(-\frac{1}{r^2}\right)\right]$$

$$v\psi = \frac{-i\hbar}{m} A\left[\frac{ik}{r} - \frac{1}{r^2}\right]e^{ikr}$$

$$v\psi = \frac{-i\hbar}{m}\left(ik - \frac{1}{r}\right) = \lambda\psi \Rightarrow v = +\frac{\hbar k}{m}$$

velocity = part independ on posi

$$S = v|\psi|^2 = \cancel{\frac{-i\hbar}{m}(ik)}\ \frac{\hbar k}{m}(\psi\psi^*)$$

$$= \frac{\hbar k}{m}\left[\frac{A}{r}e^{ikr}\cdot\frac{A}{r}e^{-ikr}\right]$$

$$\boxed{S = \frac{\hbar k}{m}\frac{|A|^2}{r^2}}$$

II Method :- $S = \frac{-i\hbar}{2m}\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right]$

$$S = \frac{-i\hbar}{2m}\left[\frac{A}{r}e^{-ikr}\cdot A\left\{\frac{ik\,e^{ikr}}{r} - \frac{e^{ikr}}{r^2}\right\} - \frac{A}{r}e^{ikr}\cdot A\left\{\frac{-ik\,e^{-ikr}}{r} - \frac{e^{-ikr}}{r^2}\right\}\right]$$

$$S = \frac{-i\hbar}{2m}\left[\frac{A^2}{r^2}\left(ik - \frac{1}{r}\right) - \frac{A^2}{r^2}\left(-ik - \frac{1}{r}\right)\right]$$

$$= \frac{-i\hbar}{2m}\frac{|A|^2}{r^2}\left[ik - \cancel{\frac{1}{r}} + ik + \cancel{\frac{1}{r}}\right]$$

$$= \frac{-i\hbar}{\cancel{2}m}\frac{|A|^2}{r^2}\cancel{2}ik$$

$$\boxed{S = \frac{\hbar k}{m}\frac{|A|^2}{r^2}}$$

A.

$\chi \quad \zeta \quad \xi$ $\quad v = \frac{p}{m} = \frac{\hbar k}{m}$

Date :
Page No. :

Que:- A beam of mono energetic particle having speed v is discribed by the wave fun $\psi(x) = u(x) e^{ikx}$ where $u(x)$ is real fun.
This corresponds to current density

(a) $u(x) v$ (b) v (c) 0 (d) $u^2(x)$

$$\psi(x) = u(x) e^{ikx}, \quad \psi^* = u(x) e^{-ikx}$$

$$S = \frac{-i\hbar}{2m}\left[\psi^* \nabla\psi - \psi\nabla\psi^*\right]$$

$$= \frac{-i\hbar}{2m}\left[u(x)\, e^{-ikx} \cdot u(x)\, ik e^{ikx} - u(x) e^{ikx} u(x) (-ik) e^{-ikx}\right]$$

$$= \frac{\hbar k}{2m}\left[u^2(x) + u^2(x)\right]$$

$$= \frac{p v}{\cancel{2}\cancel{m}}\left[\cancel{2} u^2(x)\right] \Rightarrow \boxed{S = u^2(x) v} \quad (a)$$

II Method, $\quad v\psi = \frac{-i\hbar}{m}\frac{\partial \psi}{\partial x} = \frac{-i\hbar}{m}\frac{\partial}{\partial x}\{u(x) e^{ikx}\}$

$$v\psi = \frac{-i\hbar}{m}\left\{u(x)\, ik e^{ikx} + \frac{du}{dx} e^{ikx}\right\}$$

$$v\psi = \frac{-i\hbar}{m}\left\{ik + \frac{1}{u}\frac{du}{dx}\right\} u(x) e^{ikx}$$

$$v\psi = \frac{-i\hbar}{m}\left\{ik + \frac{1}{u}\frac{du}{dx}\right\}\psi \Rightarrow v = \frac{\hbar k}{m}$$

$p = \frac{\hbar k}{m}$

$mv = \frac{\hbar k}{m}$

$v = \frac{\hbar k}{m}$

Now, $\quad S = v|\psi|^2 = \frac{\hbar k}{m}|u(x)|^2$

$$S = v\, u^2(x) \quad \text{Ans}$$

directly $\rightarrow$
$S = v|\psi|^2$
$S = v\, u^2 e^{ikx} e^{-ikx}$
$S = v\, u^2(x)$ Ans

Note:- Wave fun can also be written in different space. If we know the wave func in position space then we can convert it into mom. space, or in K space.

$\phi(p) \longrightarrow$ w.f. in mom. space

$\phi(k) \longrightarrow$ " " K - "

$\psi(x, t) = \frac{1}{\ }\int^{+\infty} \phi(k)\, e^{(ikx - \omega t)}\, dx$

Date :

Page No. :

* If we know the w. fee^n in wave vector space w-fee^n in position space,

Remember

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{ikx}\, dk$$

* In ~~mom~~ k-space, $\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi^*(x) e^{-ikx}\, dx$

* In mom. space - $\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi^*(x) e^{-i\frac{p}{\hbar}x}\, dx$

$[p = \hbar k]$

* If wave fee^n is Normalise in 1 state then will be normalise in another state. i.e.

⟶ If w.fee^n is normalise to unity in mom-sp then corresponding to position space it will normalise.

But Normalization factor may be different.

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int \psi^* e^{-ikx}\, dx$$

$$\int \phi^*(k)\phi(k)\, dk = \int \left[\frac{1}{\sqrt{2\pi}} \int \psi e^{+ikx} dx\right] \cdot \frac{1}{\sqrt{2\pi}} \int \psi e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int \phi^*(k) \int \frac{1}{\sqrt{2\pi}} \psi(x) e^{-ikx}\, dk\, dx$$

$$= \int \psi(x) \left(\frac{1}{\sqrt{2\pi}} \int \phi^*(k) e^{-ikx} dk\right) dx$$

$$= \int \psi(x) \psi^*(x)\, dx$$

$$= \int \psi^* \psi\, dx$$

If $\int \phi^*(k)\phi(k)\, dk = 1$ then $\int \psi^*(x)\psi(x)\, dx = 1$

i.e. if w.fee^n is normalise in 1 space then it will be normalise in another space.

$$\langle x \rangle = \frac{\int \psi^*(x)\, x\, \psi(x)\, dx}{\int \psi^*(x)\psi(x)\, dx} \quad \& \quad \langle p \rangle = \frac{\int \psi^*(x)\left(-i\hbar \frac{\partial}{\partial x}\right)\psi(x)\, dx}{\int \psi^*(x)\psi(x)\, dx}$$

- If we know the mom. space wave func^n then the expectation value of mom.

$$\langle p \rangle = \frac{\int \phi^*(p)\, \hat{p}\, \phi(p)\, dp}{\int \phi^*(p)\, \phi(p)\, dp}$$

As mom. operator in position space $\hat{p} = -i\hbar\frac{\partial}{\partial x}$, $\hat{x} =$
llly, position " " mom. space $\hat{x} = i\hbar\frac{\partial}{\partial p}$, $\hat{p}$.

$$K.E. = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \quad \text{(position space)}$$

$$\text{llly}, \; K.E. = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial p^2} \quad \text{(mom. space)}$$

Problem :- Find the wave func^n in wave vector space $\phi(k)$ for square wave packet

$$\psi_0(x) = \begin{cases} Ae^{ik_0 x}, & |x| \le a \quad (-a < x < +a) \\ 0, & |x| > a \quad \text{(otherwise)} \end{cases}$$

Find the factor A so that $\psi(x)$ is normalised to unity.

$A = ?$

$$\int \psi^*(x)\psi(x)\, dx = 1$$

$$\int_{-a}^{+a} Ae^{-ik_0 x}\, Ae^{ik_0 x}\, dx = 1 \Rightarrow |A|^2 \int_{-a}^{+a} e^0\, dx = 1$$

$$|A|^2 \left[x\right]_{-a}^{a} = 1 \Rightarrow |A|^2 [a + a] = 1$$

$$|A|^2 = \frac{1}{2a} \Rightarrow |A| = \frac{1}{\sqrt{2a}}$$

$$\therefore \psi_0(x) = \frac{1}{\sqrt{2a}} e^{ik_0 x}$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int \psi_0(x) e^{-ikx}\, dx$$

Date :
Page No. :

$$\phi(K) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{+a} \frac{1}{\sqrt{2a}} e^{iK_0 x} e^{-iK x} dx$$

$$= \frac{1}{2\sqrt{\pi a}} \int_{-a}^{+a} e^{i(K_0 - K)x} dx$$

$$= \frac{1}{2\sqrt{\pi a}} \left[\frac{e^{i(K_0-K)x}}{i(K_0-K)} \right]_{-a}^{+a}$$

$$= \frac{1}{2\sqrt{\pi a}} \frac{1}{i(K_0-K)} \left[e^{i(K_0-K)a} - e^{-i(K_0-K)a} \right]$$

$$= \frac{1}{\sqrt{\pi a}} \frac{1}{(K-K_0)} \left[\frac{e^{i(K-K_0)a} - e^{-i(K-K_0)a}}{2i} \right]$$

$$\boxed{\phi(K) = \frac{1}{\sqrt{\pi a}} \frac{\sin(K-K_0)}{(K-K_0)}}$$

Prob 1- Suppose that the wave fn of a particle is gi by $\psi(x) = A e^{\frac{i p_0 x}{\hbar}}$, find the $\phi(P)$.

$$\psi(x) = A e^{\frac{i p_0 x}{\hbar}}$$

$$\phi(P) = \frac{1}{\sqrt{2\pi\hbar}} \int \psi(x) e^{-\frac{i p}{\hbar} x} dx$$

$$\phi(P) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} A e^{\frac{i p_0 x}{\hbar}} e^{-i\frac{p}{\hbar} x} dx$$

$$= \frac{A}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{i(p_0 - p)\frac{x}{\hbar}} dx$$

$$= \frac{A}{\sqrt{2\pi\hbar}} \cancel{\left[\frac{e^{i(p_0-p)\frac{x}{\hbar}}}{i(p_0-p)/\hbar} \right]}$$

(?) $$\delta(x-a) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{\pm i k (x-a)} dk$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\pm i k x} dk$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \cdot \frac{A}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{i(p_0-p)\frac{x}{\hbar}} dx$$

$$\phi(P) = \frac{\sqrt{2\pi}}{\sqrt{\hbar}} A \; \delta\left(\frac{p_0 - p}{\hbar}\right)$$

$$\begin{cases} \delta(ax) = \frac{1}{a}\delta(x) \\ \delta\left(\frac{x}{a}\right) = a\,\delta(x) \end{cases}$$

$$= \sqrt{\frac{2\pi}{\hbar}} A \, \hbar \, \delta(p_0 - p)$$

$$\boxed{\phi(P) = A\sqrt{2\pi\hbar}\;\delta(p_0 - p)}$$

→ If we know the w-fu^n, then we can calculate the uncertainity product.

→ If $\psi(x)$ is plane wave type in position space then $\phi(P)$ will be dirac-delta type in mom. space & Vice-versa.

$$\delta(p - p_0) = \infty \quad , p = p_0$$
$$= 0 \quad ; p \neq p_0$$

Only possible value of p is p_0.

Prob (W.T) : Suppose the wave fu^n of a particle at a given ins-ant is $\psi(x) = A\cos kx$. find the mom. distribution fu^n $\phi(P)$. What are the possible value of mom. in this state.

If $\phi(P)$ is not asked then

$-i\hbar\nabla\psi = -i\hbar \frac{d}{dx}\psi(x) = ?$ to find the possible value of P

$$P\psi = \lambda\psi$$

$$\psi(x) = A\cos kx$$

$$\phi(P) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \psi(x) e^{-i\frac{p}{\hbar}x} dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} A\cos kx \, e^{-i\frac{p}{\hbar}x} dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} A\left(\frac{e^{ikx} + e^{-ikx}}{2}\right) e^{-i\frac{p}{\hbar}x} dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{A}{2} \int_{-\infty}^{+\infty} e^{ix\left(k - \frac{p}{\hbar}\right)} + e^{-ix\left(k + \frac{p}{\hbar}\right)} dx$$

$\delta\frac{x}{a} = \frac{a}{c}\delta(x)$

Date:
Page No.:

(?) $= \frac{\sqrt{2\pi}}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi\hbar}}\frac{A}{2}\left[\delta\left(k-\frac{p}{\hbar}\right) + \delta\left(k+\frac{p}{\hbar}\right)\right]$

$= \frac{A}{2}\sqrt{2\pi\hbar}\left[\delta(\hbar k - p) + \delta(\hbar k + p)\right]$

There are 2 possible value of mom (P).

$\hbar k - P = 0$ & $\hbar k + P = 0$

$P = \hbar k$ $\qquad P = -\hbar k$

$\boxed{P = \pm\hbar k}$

for $+x$ dir^n propogation $P = \hbar k$

$-x$ $\qquad P = -\hbar k$

→ Energy E. Value will be degenerate for e^{ikx} & e^{-ikx}.

Q4; The wave fun^n of a particle $\psi(x) = \frac{2}{a^{3/2}} x e^{-x/a}$,

$= 0$, $x \leq$

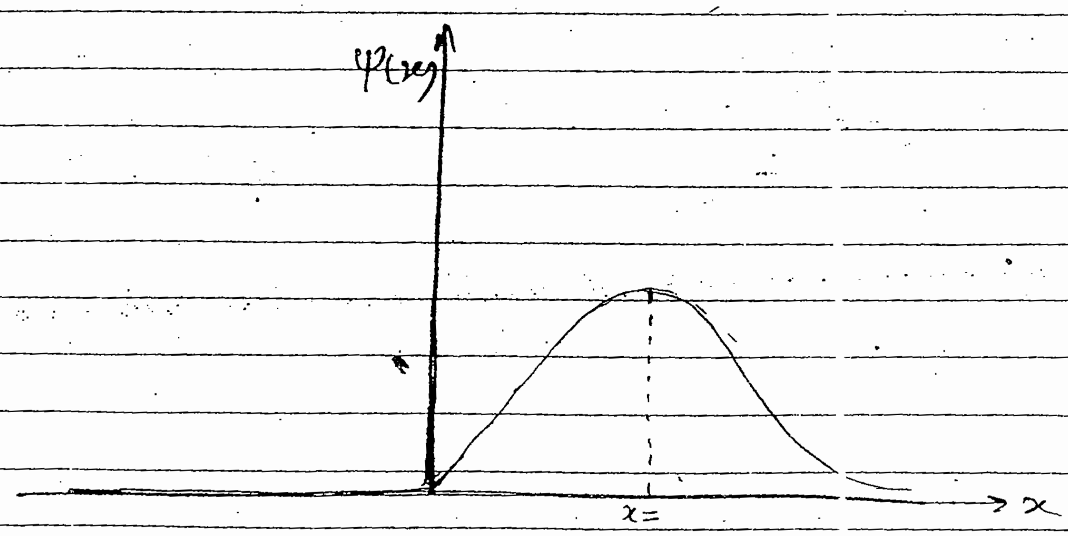
(?)

(1) sketch the wave fun^n (?)

(2) find $\phi(P)$ & $|\phi(P)|^2$ ✓ $\qquad \frac{d}{dP}|\phi(P)|^2 = 0$

(3) find the most probable value of mom.

(1)



$\psi(x)$ is ~~comp~~ product of 2 wave fun^n → one ↑
& other is ↓.

$\frac{d}{dx}|\psi|^2 = 0$ give the value of x.........

Date :
Page No. : 5

i.e. It gives the most probable value of x.

(2) $\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_0^\infty \psi(x) e^{-i\frac{p}{\hbar}x} dx$

why $\int_{-\infty}^{0} =$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_0^\infty \frac{2}{a^{3/2}} x e^{-x/a} e^{-i\frac{p}{\hbar}x} dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{2}{a^{3/2}} \int_0^\infty x e^{-\left(\frac{1}{a}+i\frac{p}{\hbar}\right)x} dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{2}{a^{3/2}} \left[\left\{ x \frac{e^{-\left(\frac{1}{a}+i\frac{p}{\hbar}\right)x}}{-\left(\frac{1}{a}+i\frac{p}{\hbar}\right)} \right\}_0^\infty - \int_0^\infty 1 \cdot \frac{e^{-\left(\frac{1}{a}+i\frac{p}{\hbar}\right)x}}{-\left(\frac{1}{a}+i\frac{p}{\hbar}\right)} dx \right]$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{2}{a^{3/2}} \left[0 + \frac{1}{\left(\frac{1}{a}+i\frac{p}{\hbar}\right)} \left\{ \frac{e^{-\left(\frac{1}{a}+i\frac{p}{\hbar}\right)x}}{-\left(+\frac{1}{a}+i\frac{p}{\hbar}\right)} \right\}_0^\infty \right]$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{2}{a^{3/2}} \left[\frac{-1}{\left(\frac{1}{a}+i\frac{p}{\hbar}\right)^2} \left\{ 0 - 1 \right\} \right]$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{2}{a^{3/2}} \frac{1}{\left(\frac{1}{a}+i\frac{p}{\hbar}\right)^2} = \sqrt{\frac{2}{\pi\hbar a}} \frac{1}{\cancel{a}} \frac{a^2\hbar^2}{(\hbar + iap)^2}$$

$$= \sqrt{\frac{2a\hbar}{\pi}} \frac{\hbar}{(\hbar+iap)^2}$$

$$|\phi(p)|^2 = \phi(p)\phi^*(p) = \frac{2a\hbar}{\pi} \frac{\hbar^2}{(\hbar+iap)^2(\hbar-iap)^2}$$

$$= \frac{2a\hbar^3}{\pi(\hbar^2+a^2p^2)^2}$$

(3) $\frac{d}{dp}\left(|\phi(p)|^2\right) = \frac{d}{dp} \frac{2a\hbar^3}{\pi(\hbar^2+a^2p^2)^2} = 0$

$$\Rightarrow \frac{2a\hbar^3}{\pi} \frac{-2(2a^2p)}{(\hbar^2+a^2p^2)^3} = 0 \quad \Rightarrow p = 0$$

Most probable value of $p = 0$

Date :
Page No. :

Bound State Problems :-

In Classical Mech,

E = -ve then state will be bound state.

But In Q.M, if E = -ve then state may but it is not necessary. In Q.M. to fin weather the state is bound or not, we use time independent Schr eqn.

If particle is completely free then the no force on particle (i.e. No restriction)

for free particle,

$$F = 0$$

$$F = -\nabla V \Rightarrow -\nabla V = 0$$

$$\frac{dV}{dx} = 0 \Rightarrow \begin{matrix} V = 0 \\ V = V_0 \end{matrix}$$

P.E is not a funcn of position & mom (it is 0 for free particle, potn or potn energy is indep of position.

Schroedinger eqn for free particle

$$H\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi$$ (When

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0$$

$$\boxed{\frac{d^2\psi}{dx^2} + k^2\psi = 0}$$ $k^2 = $

Solution, $m^2 + k^2 = 0$

roots $m = \pm ik$ (imaginary root)

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

Now if $E < V_0$ (When V

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + V_0\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V_0)\psi = 0$$

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0 \qquad , k^2 = \frac{2m(E-V_0)}{\hbar^2}.$$

or $$\frac{d^2\psi}{dx^2} - k^2\psi = 0 \qquad k^2 = \frac{2m(V_0-E)}{\hbar^2}$$

for this $m = \pm k$ (real)

$$\psi(x) = Ae^{kx} + Be^{-kx} \qquad (E < V_0)$$

for $k^2 = \frac{2mE}{\hbar^2} \Rightarrow E = \frac{\hbar^2 k^2}{2m}$

$$k^2 = \frac{2m}{\hbar^2}(V_0 - E) \Rightarrow E = \frac{\hbar^2 k^2}{2m} + V_0$$

Energy level & Pot^n shifted by a constant amount.

* To check weather the state is bound or unbound we have to compare energy & potential

If $E > V(r) \Rightarrow$ Unbound

$\boxed{E < V(r)} \Rightarrow$ bound

※ for $E < V_0 \Rightarrow$ We get +ve & -ve decaying fu^n & roots are real.

$$\psi(x) = Ae^{kx} + Be^{-kx}$$

Ae^{kx} ↓ acceptable for -ve value of K; Be^{-kx} ↓ for +ve value of K. (both decaying fu^n)

※ for $E > V_0 \Rightarrow$ roots are imaginary. unBound

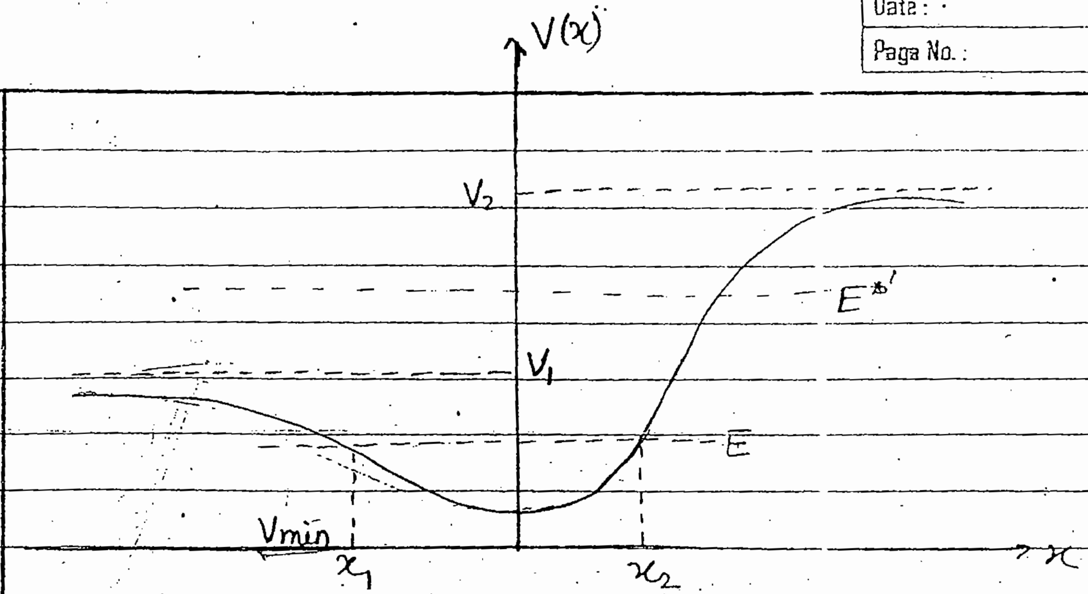
• for Conservative system; total energy of system is fixed.

$$E = K.E. + P.E.$$

• If P.E. $V = 0$ & total Energy $\underbrace{=}_{\uparrow E > V_0}$ +ve, then Unbound

$V = 0$ & " " $\underbrace{=}_{E < V_0}$ -ve, then bound

(2)



→ $V_{min} < E < V_1$, ~~for~~ ~~free~~ particle will be free b/w x_1 & x_2 but particle is not free in region

$$\left.\begin{array}{l} -\infty < x < x_1 \\ x_2 < x < +\infty \end{array}\right\} E < V_0$$

for these regions, state will be Bound Sta[te] & energy spectrum will be discrete.

→ for $E > V_2$: No restriction on particle it can move from $-\infty$ to $+\infty$. & Roots are Imaginary $e^{ik.x}$ & e^{-ikx}

for these two wave fun^n energy will be sa[me] So Energy spectrum will be doubly degene[rate] & continous.

→ If E lies b/w V_1 & V_2.

$E > V_1$, for -ve value of x, particle is free to move from $-\infty$, N[o] restriction on particle (unbound)

$E < V_1$ for +ve value of x, particle is bounded. & energy is descrete.

We get the mixture of bound & unbound state.

Potential Step :-

In potⁿ step, potⁿ will be discontinous at 1 point & discontinuty is by the finite amount ~~unity~~.

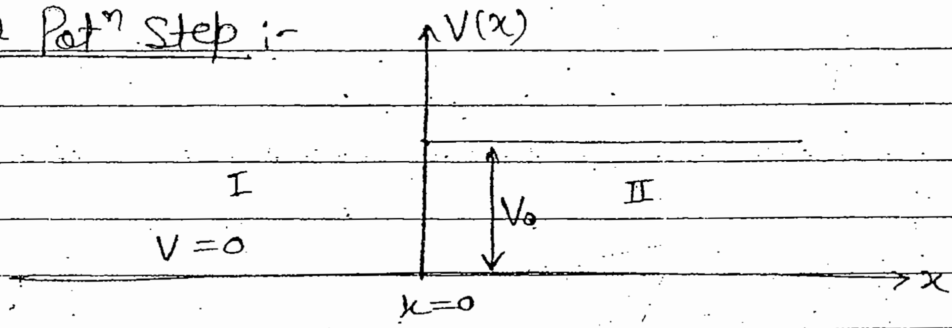
$$V(x) = \begin{cases} 0 & , x<0 \\ V_0 & , x \geq 0 \end{cases}$$

So potⁿ step is discontinous by the amount V_0.
In potⁿ step $V(x)$ is Non zero for large space.

Potential Barrier :- In potⁿ barrier, potⁿ $V(x)$ is non zero for limited region.
so there are more than 1 discontinous jump one at $x=0$ & another at $x=L$.

$$V(x) = V_0 \quad , \quad 0<x<L$$
$$= 0 \qquad \text{otherwise.}$$

Plot of Potⁿ Step :-



There are 2 different value of potⁿ so there are 2 different regions.
If we incident a beam of particles then there will be reflection & transmission due to potⁿ.
The reason of reflection is change in potⁿ.

In Scattering state or Unbound state → No need to calculate the normalization const.

but in bound state → We need to calculate "

Date :
Page No. :

Time independent 1 Dim Sch^r eq^n,

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right]\psi(x) = E\psi(x)$$

$$\Rightarrow \frac{d^2\psi}{dx^2}+\frac{2m}{\hbar^2}[E-V(x)]\psi(x)=0$$

In Region I, pot^n $V=0$

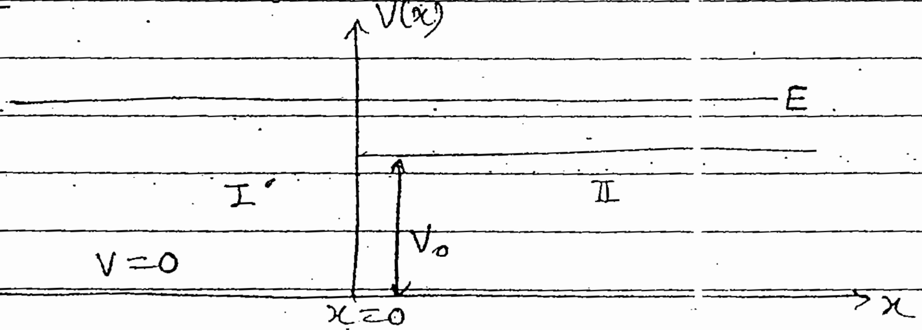
$$\frac{d^2\psi_I}{dx^2}+\frac{2m}{\hbar^2}E\psi_I=0 \qquad K_1^2=\frac{2mE}{\hbar^2}$$

In Region II, pot^n $V=V_0$

$$\frac{d^2\psi_{II}}{dx^2}+\frac{2m}{\hbar^2}(E-V_0)\psi_{II}=0 \qquad K_2^2=\frac{2m}{\hbar^2}(E$$

Solution of Sch^r eq^n in different regions :-

Case. I :- $E>V_0$



$$\frac{d^2\psi_I}{dx^2}+K_1^2\psi_I(x)=0$$

$$\frac{d^2\psi_{II}}{dx^2}+K_2^2\psi_{II}(x)=0$$

In both regions roots will be imaginary. The sol^n are

$$\psi_I = Ae^{iK_1x}+Be^{-iK_1x}$$

$$\psi_{II} = Ce^{iK_2x}+De^{-iK_2x}$$

These wave fun^{n}s ~~will~~ should satisfy all the cond^{n}s of

At boundry $x=0$, ψ_I must be same as ψ_{II}. &
" " " , derivative of I & II wave func^n's are equal.

→ $e^{ik_1 x}$ → plane wave propogating along $+x$ dir^n
i.e. incident wave

→ $e^{-ik_1 x}$ → " " " " $-x$ dir^n
reflected wave

→ $e^{ik_2 x}$ → incident wave , $e^{-ik_2 x}$ → ~~transmitted~~ ref. wave

→ There is no chance of reflection in II region
(in II region there is no reflection - only transmission)
So $D=0$

Wave func^n in IInd region,

$$\psi_{II} = Ce^{ik_2 x}$$ (Transmitted wave func^n)

Reflection Cofficient $R = \frac{|S_R|}{|S_I|}$

Transmission Cofficent $T = \frac{|S_T|}{|S_I|}$

So we have to calculate current density.

Incident current density $S_I = \frac{\hbar k_1}{m}|A|^2$

Reflection " " $S_R = \frac{-\hbar k_1}{m}|B|^2$

Transmitted " " $S_{Tr} = \frac{\hbar k_2}{m}|C|^2$

Now on putting the value of S_I, S_R, S_T, we get R & T but these are not actual value of

Date :
Page No. :

R & T bcoz A, B, C are arbitary constan
So we have to eleminate these constants. To eleminate them use boundry conditions –

At $x=0$, $\psi_I = \psi_{II}$

$$\frac{d\psi_I}{dx} = \frac{d\psi_{II}}{dx}$$

There are 2 B.C. & unknown contants are 3
So we determine A & B in terms of C.
These cond^{n}s implies that

$$A + B = C \quad \text{———} (1)$$

$$iK_1(A - B) = iK_2C \quad \text{———} (2)$$

$K_2 \times (1) - (2) \Rightarrow \quad K_2(A+B) = K_2C$

$$\cancel{i}K_1.(A-B) = \cancel{i}K_2C$$

$$(K_2 - K_1)A - (K_1+K_2)B = 0$$

$$\Rightarrow (K_1 - K_2)A = (K_1+K_2)B$$

So $R = \dfrac{|B|^2}{|A|^2} = \dfrac{|K_1 \mp K_2|^2}{|K_1+K_2|^2}$ $\left\{ \begin{array}{l} eq^n(1) \times K_1 + eq^n(2) \times \\ \Rightarrow \\ C = \left(\dfrac{2K_1}{K_1+K_2}\right) A \end{array} \right.$

$T = \dfrac{K_2}{K_1} \dfrac{|C|^2}{|A|^2} = \dfrac{4K_1K_2}{(K_1+K_2)^2}$

We can write $R = \dfrac{(K_1-K_2)^2}{(K_1+K_2)^2}$ as K_1 & K_2 are real.

In terms of mom. $(p = \hbar k)$

$$R = \frac{(p_1-p_2)^2}{(p_1+p_2)^2}$$

$$T = \frac{4p_1p_2}{(p_1+p_2)^2}$$

In terms of energy,

$$R = \left(\frac{\sqrt{\frac{2mE}{\hbar^2}} - \sqrt{\frac{2m}{\hbar^2}(E-V_0)}}{\sqrt{\frac{2mE}{\hbar^2}} + \sqrt{\frac{2m}{\hbar^2}(E-V_0)}} \right)^2$$

$$\boxed{R = \left(\frac{\sqrt{E} - \sqrt{E-V_0}}{\sqrt{E} + \sqrt{E-V_0}} \right)^2}$$

$$\boxed{T = \frac{4\sqrt{E}\sqrt{E-V_0}}{\left(\sqrt{E} + \sqrt{E-V_0}\right)^2}}$$

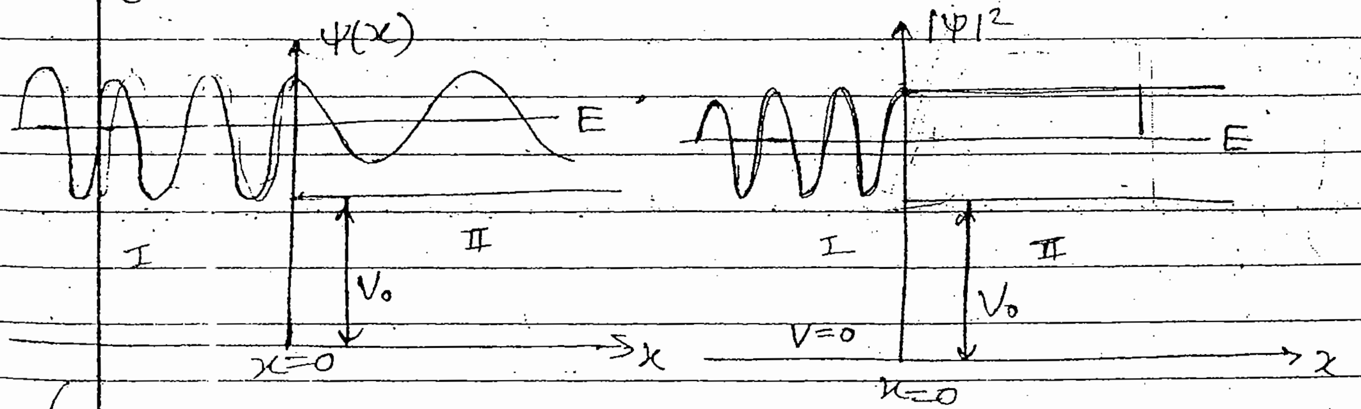
→ Q. Mechanically, there is finite chance of reflection & transmission

→ Classically, there is 100 % chance of ~~transmission~~, No reflection.

~~If~~ $E \sim V$ then there is large chance of ~~transmission~~ reflection.

$\boxed{R + T = 1}$ No absorbtion

{ $R + T < 1 \Rightarrow$ if There is absorption }



→ In I Region - We can eisely plot the wave fun^n $\psi = Ae^{ik_1x} + Be^{-ik_1x}$, have both reflection part & incident part. In II region - amp. & W.L. will change

$p = \pi k$

Date :
Page No. :

In II region – $\psi = Ce^{ik_2 x}$

We can only plot the real part of this func^n. for imaginary part plot will be compl

In I Region, Total Energy $E_1 = (K.E.)_I$

In II Region, $E_2 = (K.E.)_{II} + V_0$

→ K.E. for IInd region < K.E. of Ist region bec Total energy E is constant.

i.e. $(K.E.)_{II} < (K.E.)_I$

$$\boxed{p_2 < p_1}$$

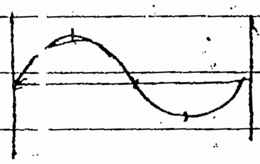
$$\boxed{\lambda_2 > \lambda_1} \quad \text{or} \quad f_2 < f_1$$

→ Prob. of finding the particle in I region (for I plot) is same. & in II region it is constant

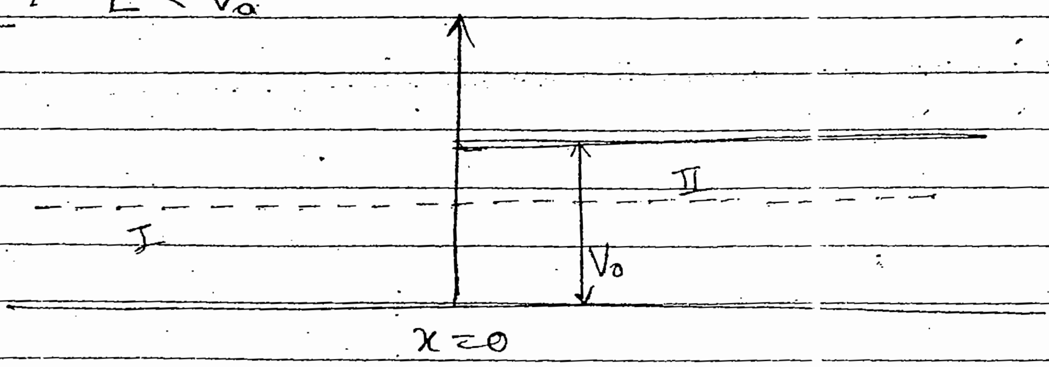
$$\psi_{II} = Ce^{ik_2 x}$$

$$|\psi_{II}|^2 = C^2 \quad \text{(constant.)}$$

Note :– At all the points, amplitude is same – Standing wave



Case. II :– $E < V_0$



for I – region, There is No change in W. func^n

$$\psi_I = Ae^{ik_1 x} + Be^{-ik_1 x}$$

In II Region, $\frac{d^2\psi_{II}}{dx^2} + \frac{2m}{\hbar^2}(E-V_0)\psi_{II}(x) = 0$

$$K_2^2 = \frac{2m}{\hbar^2}(V_0 - E)$$

$$\Rightarrow \frac{d^2\psi_{II}}{dx^2} - k^2\psi_{II}(x) = 0 \quad \text{(roots will be real)}$$

Solution is, $\psi_{II} = Ce^{K_2 x} + De^{-K_2 x}$ {This is not rep^n plane wave, it is decay fuc^n}

Wave fuc^n must be finite but at $x = \infty$, $e^{K_2 x} = \infty$ So ψ_{II} becoms ∞ So This w. fuc^n ~~mu~~ should be modified to make the w. fuc^n finite.

$$\boxed{C = 0}$$

{but at $x = -\infty$, $D = 0$ but -ve values of x are not allowed in IInd region}

$$\boxed{\psi_{II} = De^{-K_2 x}}$$

This state rep^n the bound state. Its prob. of transmission is decreasing

$x \uparrow$ then $|\psi|^2$ will $\downarrow$.

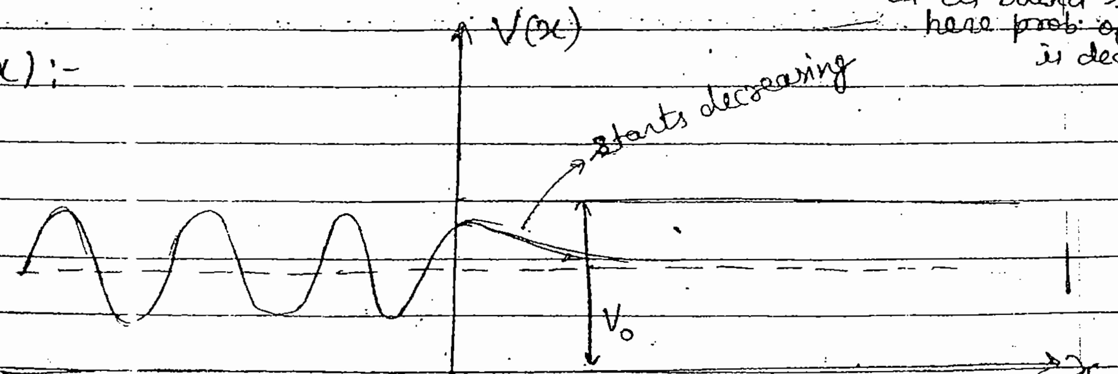
As $x \to \infty$, $|\psi|^2 \to 0$

$$\left\{ |\psi|^2 = |D|^2 e^{-2K_2 x} \right\}$$

There is no chance that particle found at $x = \infty$.

We'll not calculate R & T. (We can but no need) bcoz it is bound state & here prob. of trans. is decaying.

$\psi(x)$:-



L → ••, energy less so can not cross the barrier → N

Date :
Page No. :

Classically → There is 100% chance of reflection
Q. Mechanically → finite chance of reflection & tra

If $E = V_0$ then $K_2 = 0$, $\psi_{II} = ?$
so we'll get a straight line.

→ $E < V_0$ region is Not classically alowed.
bcoz $E = K.E. + V_0$
if $K.E. = 0$ then $E = V_0$
but never less than V_0.
It is Classically forbidden region.
→ But In Quantum mechanics, $E < V_0$ regio is allowed.

→ Phenomenon in which finite chance of to fo the particle in classically forbidden regio is called Tunnelling.
In case of potⁿ barrier – Barrier Penetration

Particle in a Box :-
OR
Infinite potⁿ Well :- There are 2 types of infi potⁿ well.

1) Unsymmetric   2) symmetric

1) Unsymmetric potⁿ well :- $f(x) = -f(x)$

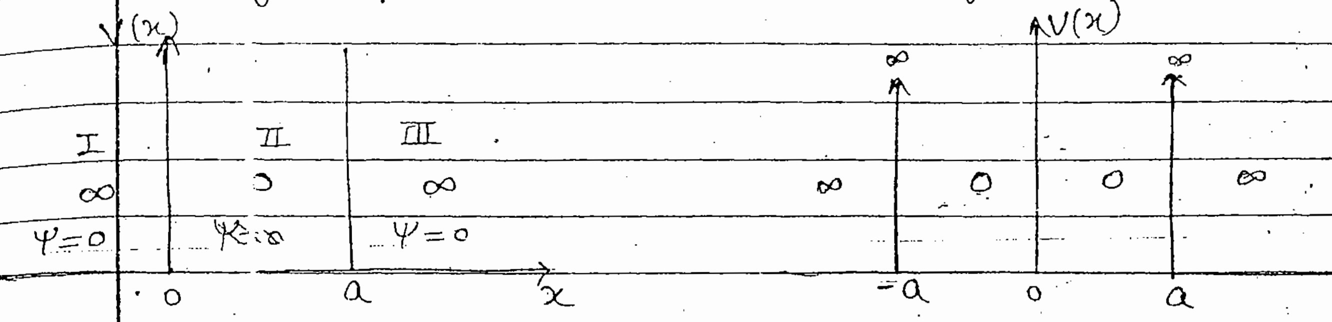
$$V(x) = \begin{cases} 0 & , 0 < x < a \\ \infty & , \text{otherwise} \end{cases}$$

2) Symmetric :- $f(x) = f(-x)$ then funcⁿ is symmetric.

$$V(x) = \begin{cases} 0 & , -a < x < a \\ \infty & , \text{otherwise} \end{cases}$$

If $f(-x) \neq \pm f(+x)$ then it is neither even nor odd.

Unsymmetric   Symmetric.



Unsymmetric Pot^n well :-

If pot^n is ∞, width is ∞ then it can not penetrate the wall. so particle can not found outside the box. So outside the box $\psi = 0$.

Sch^r eq^n inside the well,

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0$$

$$K^2 = \frac{2mE}{\hbar^2}$$

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0$$

Its solution,
$$\left.\begin{aligned} \psi &= Ae^{ikx} + Be^{-ikx} \\ \psi &= A'\sin kx + B'\cos kx \end{aligned}\right\} \text{same}$$

$$\left\{\begin{aligned} \psi &= A(\cos kx + i\sin kx) + B(\cos kx - i\sin kx) \\ &= (A+B)\cos kx + i(A-B)\sin kx \\ &= B'\cos Kx + A'\sin kx \end{aligned}\right\}$$

There are 2 boundries so 2 discontinous jumps
→ at $x = 0$ & $x = a$.
So 2 B.C.s

At $x = 0$, $\psi_I = \psi_{II} \Rightarrow \psi_{II} = 0$

At $x = a$, $\psi_{II} = \psi_{III} \Rightarrow \psi_{II} = 0$

i.e. $\psi(0) = 0 = \psi(a)$

Date :
Page No. :

At $x=0$, $\psi=0$

$$0 = A\sin(k\times 0) + B\cos 0$$

$$\Rightarrow B = 0$$

$$\Rightarrow \psi(x) = A\sin kx$$

Now at $x=a$, $\psi=0$

$$\Rightarrow A\sin ka = 0$$

$$\sin ka = 0 \quad , \quad A \neq 0$$

A can never be zero so $\sin ka = 0$ (always) bcoz if $A=0$ then particle can not exist.

$$\Rightarrow ka = n\pi \quad , \quad n = ,2,3,4,\ldots$$

$$\boxed{k = \frac{n\pi}{a}} \quad \Rightarrow \psi(x) = A\sin\left(\frac{n\pi x}{a}\right)$$

~~for completely free particle no restriction on k~~ There is restriction on the value of k. The value of k is not continous, it is discrete.

$$k^2 = \frac{2mE}{\hbar^2}$$

$$E = \frac{\hbar^2 k^2}{2m} \qquad \left(k = \frac{n\pi}{a}\right)$$

$$\boxed{E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}} \qquad \text{for } n\text{th state.}$$

We know $k = \frac{2\pi}{\lambda}$

$$\Rightarrow \lambda = \frac{2\pi}{k} = \frac{2\pi a}{n\pi} = \frac{2a}{n}$$

$$\boxed{\lambda = \frac{2a}{n}}$$

If $n\uparrow$, $\lambda\downarrow$ & $E\uparrow$

Normalization Cond^n:-

$$\int_{\text{all space}} \psi_n^* \psi_n \, dx = 1.$$

$$\int_0^a |A|^2 \sin^2\left(\frac{n\pi x}{a}\right) dx = 1 \Rightarrow |A|^2 \int_0^a \frac{\left(1-\cos\frac{2n\pi x}{a}\right)}{2}\, dx = 1$$

$$\Rightarrow \frac{|A|^2}{2}\left[x - \left(\frac{a}{2n\pi}\right)\sin\frac{2n\pi x}{a}\right]_0^a = 1$$

$$\Rightarrow \frac{|A|^2}{2}[a] = 1 \Rightarrow |A|^2 = \frac{2}{a}$$

$$\Rightarrow A = \sqrt{\frac{2}{a}} = \sqrt{\frac{2}{\text{width}}} \Rightarrow \boxed{\psi(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)}$$

So $E_n = \dfrac{n^2\pi^2\hbar^2}{2m(\text{width})^2}$ & $\lambda = \dfrac{2\times \text{width}}{n}$

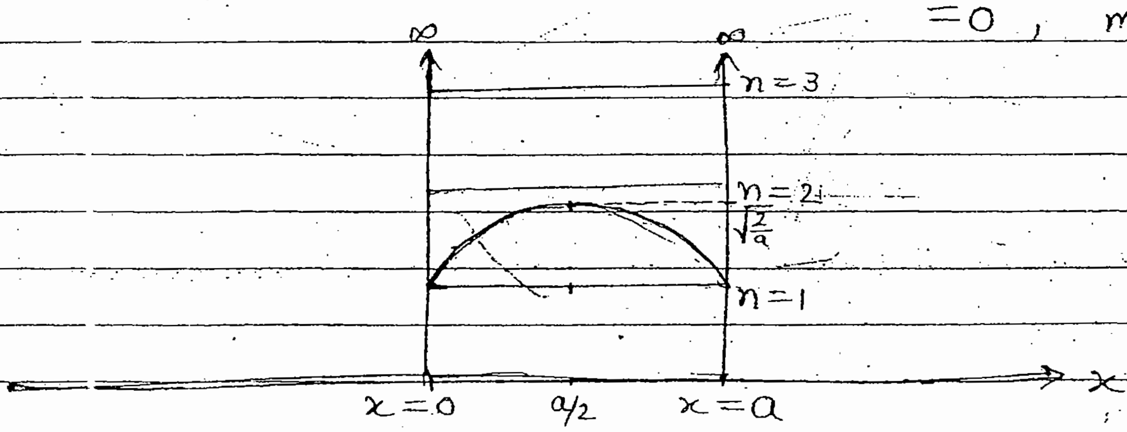
Also $\psi_n(x) = \sqrt{\dfrac{2}{\text{width}}}\sin\left(\dfrac{n\pi x}{\text{width}}\right)$

Mom: $p = \hbar k$ in +ve x dirⁿ
$p = -\hbar k$ in −ve x dirⁿ

So possible value of mom: $p = \pm \hbar k$

$p = \pm \dfrac{n\pi\hbar}{a}$

Orthonormality Condⁿ:- $\int \psi_m^*(x)\,\psi_n(x)\,dx = \delta_{mn}$
$= 1, \quad m = n$
$= 0, \quad m \neq n$



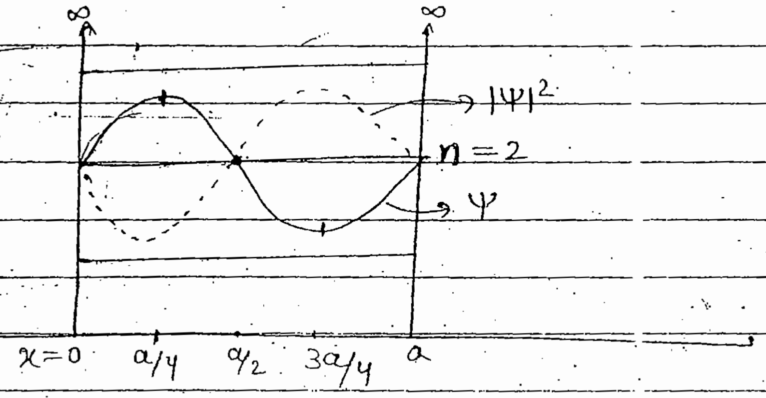
for $n=1$, $\psi_1(x) = \sqrt{\dfrac{2}{a}}\sin\left(\dfrac{\pi x}{a}\right)$

at $x=0$, $\psi_1(x) = 0$, at $x = \frac{a}{2}$, sin → max, at $\pi/2$
$x = a$ $\psi_1(x) = 0$ $\psi_1(x) = \sqrt{\frac{2}{a}}$

At $n=2$, $\psi_2 = \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right)$, at a, $\psi_2 = 0$

at $x = \frac{a}{4}$, $\psi_2 = \sqrt{\frac{2}{a}}$ ~~(maxi value)~~

$= \frac{3a}{4}$, $\psi_2 = -\sqrt{\frac{2}{a}}$ (min. value)

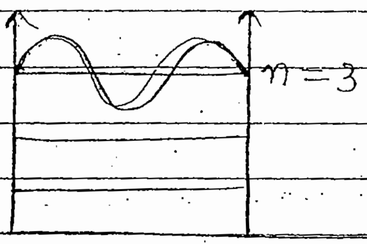


<u>Nodes</u> → ~~At which wave fuc^n = 0~~

for nth state : No. of nodes(~~at~~ including boundry) = $n+1$

" " (excluding boundry) = $n-1$

<u>for $n=3$</u>,



<u>Most probable value of position</u> :- The value at with probability density is maximum ~~as~~ is the most prob. value of position.

$$\frac{d}{dx} |\psi_x|^2 = 0 \Rightarrow x = ?$$

$$\frac{d^2}{dx^2} |\psi_x|^2 < 0$$

If there is more than 1 value of x, then we have to check the IInd derivative to find the most probable value of x.

* What is the prob. of particle in n^{th} state for $0 < x < a/2$.

$$\psi_n = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$$

$$\cancel{= \frac{2}{a} \int \sin}$$

$$\int \psi_n^* \psi_n \, dx = \int_0^{a/2} \left(\sqrt{\frac{2}{a}}\right)^2 \sin^2\left(\frac{n\pi x}{a}\right) dx$$

for $n = 1$,

$$= \int_0^{a/2} \frac{2}{a} \frac{1}{2} \left(1 - \cos \frac{2\pi x}{a}\right) dx$$

$$= \frac{1}{a} \int_0^{a/2} \left[1 - \cos \frac{2\pi x}{a}\right] dx$$

$$= \frac{1}{a} \left[x - \frac{\sin \frac{2\pi x}{a}}{2\pi/a}\right]_0^{a/2}$$

$$= \frac{1}{a}\left(\frac{a}{2}\right) = \frac{1}{2}$$

for $n = 2$, $= \frac{1}{2}$

for $n = 3$, $= \frac{1}{2}$

$$\langle x \rangle = \frac{\int_0^a \psi^* x \psi \, dx}{\int_0^a \psi^* \psi \, dx} = \int_0^a \psi^* x \psi \, dx$$

$$= \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) x \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{2}{a} \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) x \, dx$$

$$= \frac{1}{2} \frac{2}{a} \int_0^a x \left[1 - \cos \frac{2n\pi x}{a}\right] dx$$

$$= \frac{1}{a} \left[\frac{x^2}{2} - \frac{a}{2n\pi} x \frac{\sin \frac{2n\pi x}{a}}{2n\pi/a} - \cos \frac{2n\pi x}{a}\right]_0^a$$

$$= \frac{1}{a}\left[\frac{a^2}{2} - \frac{a}{2n\pi} \sin 2n\pi - \cos 2n\pi + 1\right] = \frac{1}{a}\left[\frac{a^2}{2} - 0 - 1 + 1\right]$$

$$= a/2$$

Date :
Page No. :

$$\langle p \rangle = \int_0^a \psi^* \left(-i\hbar \frac{\partial}{\partial x}\right) \psi \, dx$$

$$= \int_0^a \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \left(-i\hbar \frac{\partial}{\partial x}\right) \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \, dx$$

$$= 0$$ (for back & forth motion, mom will always zero $\langle P \rangle = 0$.

$\langle x^2 \rangle = ?$

$\langle p^2 \rangle = ?$

$$\Delta x \, \Delta p = n\pi\hbar \sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}$$

$$\Delta x = a \sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}$$

$$\Delta p = \frac{n\pi\hbar}{a}$$

Now find $\langle x^2 \rangle$ & $\langle p^2 \rangle$,

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} \psi_n^*(x) \, x^2 \, \psi_n(x) \, dx$$

$$= \int_0^a \frac{2}{a} \sin \frac{n\pi x}{a} \, x^2 \sin \frac{n\pi x}{a} \, dx$$

$$= \frac{2}{a} \int_0^a x^2 \sin^2 \frac{n\pi x}{a} \, dx$$

$$= \frac{2}{a} \int_0^a \frac{x^2}{2} \left(1 - \cos \frac{2n\pi x}{a}\right) dx$$

$$= \frac{1}{a} \left[\left[\frac{x^3}{3}\right]_0^a - \left(x^2 \frac{\sin \frac{2n\pi x}{a}}{2n\pi/a} \right)_0^a + \int_0^a 2x \cdot \frac{\sin \frac{2n\pi x}{a}}{2n\pi/a} \right.$$

$$= \frac{1}{a} \left[\frac{a^3}{3} - \frac{a}{2n\pi}(0+0) + 2 \left\{ -x \frac{\cos \frac{2n\pi x}{a}}{\left(\frac{2n\pi}{a}\right)^2} + \frac{\sin \frac{2n\pi x}{a}}{\left(\frac{2n\pi}{a}\right)^{}} \right. \right.$$

$$= \frac{1}{a} \left[\frac{a^3}{3} - 0 - 2 \cdot a \frac{a^2}{(2n\pi)^2} \cos 2n\pi - 0 + (0-0 \right.$$

$$= \frac{1}{a} \left[\frac{a^3}{3} - \frac{a^3}{2(n\pi)^2}(1) \right]$$

$$\boxed{\langle x^2 \rangle = \frac{a^2}{3} - \frac{a^2}{2n^2\pi^2}}$$

$$\langle p_x^2 \rangle = \int_0^a \frac{2}{a} \psi^*(x)\left(-\hbar^2 \frac{d^2}{dx^2}\right)\psi(x)\,dx$$

$$= \frac{2}{a}\int_0^a \sin\frac{n\pi x}{a}\left(-\hbar^2\frac{d^2}{dx^2}\right)\sin\frac{n\pi x}{a}\,dx$$

$$= \frac{2}{a}\hbar^2\left(\frac{n\pi}{a}\right)^2\int_0^a \sin^2\frac{n\pi x}{a}\,dx$$

$$= \frac{2}{a}\hbar^2\left(\frac{n\pi}{a}\right)^2\frac{1}{2}\left[x - \frac{\sin\frac{2n\pi x}{a}}{2n\pi/a}\right]_0^a$$

$$= \frac{\cancel{2}}{\cancel{a}}\frac{\hbar^2}{\cancel{2}}\left(\frac{n\pi}{a}\right)^2(\cancel{a})$$

$$\boxed{\langle p_x^2\rangle = \frac{n^2\hbar^2\pi^2}{a^2}}$$

Now uncertainity in position,

$$\Delta x = \left[\langle x^2\rangle - \langle x\rangle^2\right]^{1/2}$$

$$= \left[\frac{a^2}{3} - \frac{a^2}{2n^2\pi^2} - \frac{a^2}{4}\right]^{1/2}$$

$$= a\left[\frac{1}{3} - \frac{1}{4} - \frac{1}{2n^2\pi^2}\right]^{1/2}$$

$$= a\left[\frac{1}{12} - \frac{1}{2n^2\pi^2}\right]^{1/2}$$

$$\boxed{\Delta x = a\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}}$$

Uncertainity in momentum,

$$\Delta p_x = \left[\langle p_x^2\rangle - \langle p_x\rangle^2\right]^{1/2}$$

$$= \left[\frac{n^2\hbar^2\pi^2}{a^2} - 0\right]^{1/2}$$

$$\boxed{\Delta p_x = \frac{n\pi\hbar}{a}}$$

Date :
Page No. :

Product, $\Delta x \Delta p_x = n\pi\hbar \sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}$

Some Imp. points :-

The wavefuncⁿ $\psi(x)$ is continuous & finite ev
unless there is an infinite jump in the
potential, its slope $\frac{d\psi}{dx}$ is also continous.

A bound state can appear if the potⁿ pa
through a min.

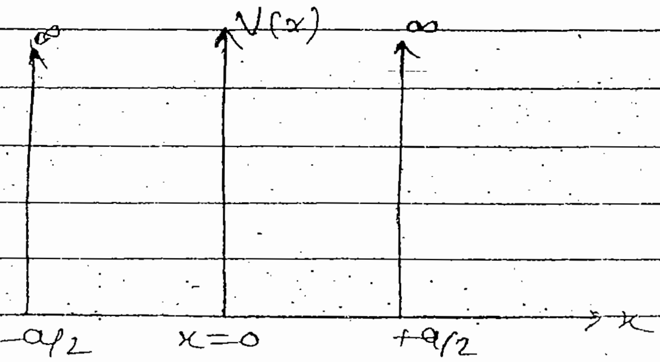
Bound state energies are discrete :- If we
measure the energy, it is either E_1, E_2 o
E_3 & so on. Classically, the energy ca
take any value but Q. mechanically not
all value are allowed. This is a gene
property of bound states in Q.M.

For any given e.f., there are certain va
of x, where the w.f. is zero. Those value
or pt. are ~~call~~ called nodes. The no. of nod
incʳ as we go for higher energy eigenfu
This is a general phenomenon in bound
states.

Symmetric Potential Well :-

$$V(x) = 0 \quad , \quad |x| < a/2 \quad [-a/2 < x < +a/2]$$
$$\qquad \infty \quad , \quad |x| \geq a/2 \quad [\text{otherwise}]$$

Sch^r eq^n is same.
The wave fuc^n in the case will be changed.
Outside the pot^n well, pot^n is ∞ & 0,
We can solve the Sch^r eq^n only inside the well:



$$\psi(x) = A \sin kx + B \cos kx$$

Boundary Cond^n :- when pot$^n \to \infty$ then derivative is discontinous.

$$\psi(-a/2) = 0$$
$$\psi(+a/2) = 0$$

$$\Rightarrow \quad A \sin \frac{Ka}{2} + B \cos \frac{Ka}{2} = 0 \quad \text{---} (1)$$
$$-A \sin \frac{Ka}{2} + B \cos \frac{Ka}{2} = 0 \quad \text{---} (2)$$

eq^n (1) + (2) $\Rightarrow \quad 2B \cos \frac{Ka}{2} = 0 \Rightarrow \frac{Ka}{2} = \frac{n\pi}{2}$

eq^n (1) − (2) $\Rightarrow \quad 2A \sin \frac{Ka}{2} = 0$

$$K = \frac{n\pi}{a} \quad , \quad n = 1, 3, 5, 7, \ldots$$

& $A \sin \frac{Ka}{2} = 0 \Rightarrow \frac{Ka}{2} = m\pi \quad , \quad m = 1, 2, 3, 4, 5 \ldots$

$$\Rightarrow K = \frac{2m\pi}{a} = \frac{n\pi}{a} \qquad n = 2, 4, 6 \ldots$$

Date :
Page No. :

General Wave fun^n,

$$\cancel{\psi(x) = A \sin kx + B \cos kx}$$

$$\psi(x) = A \sin \frac{n\pi x}{a} + B \cos \frac{n\pi x}{a}$$

for odd values of n

$$\psi(x) = \sqrt{\frac{2}{a}} \cos\left(\frac{n\pi x}{a}\right) \quad , n = 1, 3, 5 \ldots$$

for even value of n,

$$= \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \quad , n = 2, 4, 6 \ldots$$

$$K^2 = \frac{2mE}{\hbar^2} \Rightarrow \boxed{E = \frac{n^2\pi^2\hbar^2}{2ma^2}}$$

nth state wave fun^n for symmetric pot^n well is

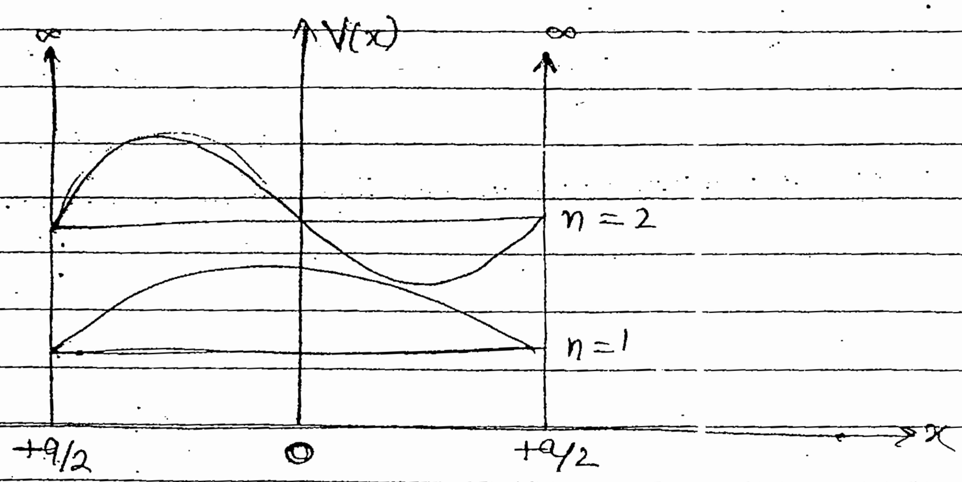
$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left[\frac{n\pi}{a}(x + a/2)\right]$$

$$\cancel{, n = 1, 2, 3, 4 \ldots}$$

Wave fun^n of unsymmetric pot^n well

$$\psi(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$$

Put $x = a/2 + x$, we get w. fun^n for symmetric P.W.



Expectation value of x, $\langle x \rangle = \int_{-a/2}^{a/2} \psi^* x \psi \, dx$

$+a/2$ to $\infty \rightarrow \psi = 0$ $\}$ $-a/2$ to $a/2$.

Date :
Page No. : 61

$$\langle x \rangle = \int_{-a/2}^{a/2} \left(\sqrt{\frac{2}{a}}\right)^2 \sin^2 \frac{n\pi}{a}\left(x+\frac{a}{2}\right) \cdot x \, dx \qquad \{\text{odd fun}^n\}$$

$$\langle x \rangle = 0$$

$$\langle p \rangle = \int_{-a/2}^{+a/2} \frac{2}{a} \sin\left[\frac{n\pi}{a}\left(x+\frac{a}{2}\right)\right]\left(-i\hbar\frac{\partial}{\partial x}\right) \sin\left[\frac{n\pi}{a}\left(x+\frac{a}{2}\right)\right] dx = 0$$

Problem :- In normalise wave fun^n of a particle in 1 Dim box extending from 0 to L is

$$\psi(x) = \begin{cases} \sqrt{\frac{12}{L^3}}\, x & \text{for } 0 < x < \frac{L}{2} \\ \sqrt{\frac{12}{L^3}}\, L-x & \text{for } \frac{L}{2} < x < L \\ 0 & \text{otherwise} \end{cases}$$

The prob. of finding the particle in ground state

(a) $\frac{98}{\pi^4}$ (b) $\frac{96}{\pi^4}$ (c) $\frac{96}{\pi^3}$ (d) $\frac{48}{\pi^3}$

$$\cancel{|\psi|^2 = \int_0^{L/2} \sqrt{\frac{12}{L^3}}\, x \cdot \sqrt{\frac{12}{L^3}}\, x \, dx + \int_{L/2}^{L} \sqrt{\frac{12}{L^3}}(L-x)\sqrt{\frac{12}{L^3}}(L-x)\, dx}$$

$$= \int_0^{L/2} \left(\frac{12}{L^3}\right) x^2 dx + \left(\frac{12}{L^3}\right)\int_{L/2}^{L} (L-x)^2 dx$$

$$= \frac{12}{L^3}\left(\frac{x^3}{3}\right)_0^{L/3} + \left(\frac{12}{L^3}\right)\left[L^2 x + \frac{x^3}{3} - 2L\frac{x^2}{2}\right]_{L/2}^{L}$$

$$= \frac{1}{3}\frac{12}{L^3}\left[\frac{L^3}{27}\right] + \frac{12}{L^3}\left[L^3 + \frac{L^3}{3} - L^3 - \frac{L^3}{2} + \frac{L^3}{8\times 3} - \frac{L^3}{4}\right]$$

Wave fun^n for the particle in 1-D box,

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

for $n = 1$, $\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right)$

Date :

Page No. :

Prob. of finding the particle in ground state is $\rightarrow |c_1|^2$

$$c_1 = \int_{-\infty}^{+\infty} \phi_1^*(x)\, \psi \, dx$$

$$c_1 = \int_0^{L/2} \phi_1^*(x)\, \psi(x)\, dx + \int_{L/2}^{L} \phi_1^*(x)\, \psi(x)\, dx +$$

$$c_1 = \int_0^{L/2} \sqrt{\frac{2}{L}} \sin\frac{\pi x}{L} \cdot \sqrt{\frac{12}{L^3}}\, x \, dx + \int_{L/2}^{L} \sqrt{\frac{2}{L}} \sin\frac{\pi x}{L} \sqrt{\frac{12}{L^3}} (L-x)$$

$$c_1 = \sqrt{\frac{24}{L^4}} \left[\int_0^{L/2} x \sin\frac{\pi x}{L}\, dx + \int_{L/2}^{L} (L-x) \sin\frac{\pi x}{L}\, dx \right]$$

$$= \sqrt{\frac{24}{L^4}} \left\{ \left[-x\cos\frac{\pi x}{L}\left(\frac{L}{\pi}\right) + \frac{\sin\frac{\pi x}{L}}{\left(\frac{\pi}{L}\right)^2} \right]_0^{L/2} + \left[-(L-x)\cos\frac{\pi x}{L} - \frac{\sin \pi x/L}{(\pi/L)^2} \right]_{L/2}^{L} \right\}$$

$$= \frac{\sqrt{24}}{L^2} \left\{ \left[0 + \frac{L^2}{\pi^2} \right] + \left[0 + \frac{L^2}{\pi^2} \right] \right\}$$

$$= \frac{\sqrt{24}}{L^2} \left[\frac{2L^2}{\pi^2} \right]$$

$$c_1 = \frac{2\sqrt{24}}{\pi^2} = \frac{\sqrt{96}}{\pi^2}$$

Prob. of finding the particle in ground state is $= |c_1|^2$

$$|c_1|^2 = \frac{96}{\pi^4} \quad \underline{\text{Ans}} \quad (b)$$

Prob :- Particle of mass m which moves freely inside an infinite ∞ potⁿ well of width a, $\left(V(x) = \begin{matrix} 0 & , 0<x<a \\ \infty & \text{otherwise} \end{matrix} \right)$

has the following wave fucⁿ at $t=0$,

$$\psi(x,0) = \frac{A}{\sqrt{a}} \sin\left(\frac{\pi x}{a}\right) + \sqrt{\frac{3}{5a}} \sin\left(\frac{3\pi x}{a}\right) + \frac{1}{\sqrt{5a}} \sin\left(\frac{5\pi x}{a}\right)$$

A is real constant.

(1) Find the value of A so that $\psi(x,0)$ is normalise to unity.

(2) If measurement of energies are carried out what are the values that will be found & what are the corresponding probabilities.
Also calculate the $\langle E \rangle$.

(3) find the wave fucⁿ at time t, $\psi(x,t)$

(4) Determine the prob. of finding the system at time t in the state,

$$\phi(x,t) = \sqrt{\frac{2}{a}} \sin\left(\frac{5\pi x}{a}\right) e^{-\frac{iE_5 t}{\hbar}}$$

Note - $\psi = \sum_n C_n \phi_n = C_1\phi_1 + C_2\phi_2 + C_3\phi_3 + \cdots$

ϕ_1, ϕ_2, ϕ_3 are orthonormal wave fucⁿs.

$$\psi^* = \sum_m C_m^* \phi_m^*$$

$$\psi^*\psi = \sum_m C_m^* \phi_m^* \sum_n C_n \phi_n$$

$$\int_{\text{all space}} |\psi|^2 dx = 1$$

$$\int \sum_m C_m^* \phi_m^* \sum_n C_n \phi_n \, dx = 1$$

$$\sum_m \sum_n C_m^* C_n \int \phi_m^* \phi_n \, dx = 1$$

Date:
Page No.:

If $\int \phi_m^* \phi_n \, dx = \delta_{mn}$

So $\sum_m \sum_n C_m^* C_n \delta_{mn} = 1$

If $m \neq n$ then $\delta_{mn} = 0$

$m = n$, $\sum_n |C_n|^2 = 1$

$$\psi = \sum_n C_n \phi_n = C_1\phi_1 + C_2\phi_2 + C_3\phi_3 + \cdots + C_n\phi_n$$

Prob. of finding the particle in ground state is $= |C_1|^2$

We have to calculate the cofficients.

$$\int \phi_n^* \psi \, dx = \int (C_1\phi_1 + C_2\phi_2 + \cdots + C_n\phi_n + \cdots) dx$$

$$\boxed{C_n = \int \phi_n^* \psi \, dx}$$

$$\boxed{P_n = |C_n|^2}$$

If a linear combination is given in w. fun^n then prob. may be differnt for each state depending upon these cofficients (C_n).

$$\sum |C_n|^2 = 1$$

Sol^n

$$\psi(x,0) = \frac{A}{\sqrt{a}} \sin\left(\frac{\pi x}{a}\right) + \sqrt{\frac{3}{5a}} \sin\left(\frac{3\pi x}{a}\right) + \frac{1}{\sqrt{5a}} \sin\left(\frac{5\pi x}{a}\right)$$

$$= \frac{\sqrt{2}}{\sqrt{a}} \frac{A}{\sqrt{2}} \sin\left(\frac{\pi x}{a}\right) + \sqrt{\frac{2}{a}} \sqrt{\frac{3}{10}} \sin\left(\frac{3\pi x}{a}\right) + \frac{\sqrt{2}}{\sqrt{a}} \sqrt{\frac{1}{10}} \sin\left(\frac{5\pi x}{a}\right)$$

Now, $\sum |C_n|^2 = 1$ $= \frac{A}{\sqrt{2}} \phi_1(x) + \sqrt{\frac{3}{10}} \phi_3(x) + \sqrt{\frac{1}{10a}} \phi_5(x)$

$$\Rightarrow \frac{A}{\sqrt{2}} \left(\sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) \right) - \sqrt{\frac{3}{10}} \left[\sqrt{\frac{2}{a}} \sin\left(\frac{3\pi x}{a}\right) \right] +$$

$$\sqrt{\frac{1}{\cdot}} \left[\sqrt{\frac{2}{\cdot}} \sin\left(\frac{5\pi x}{\cdot}\right) \right]$$

Now, $\Sigma |C_n|^2 = 1$

$$\Rightarrow \frac{A}{2} + \frac{3}{10} + \frac{1}{10} = 1$$

$$\frac{A}{2} = 1 - \frac{4}{10} \Rightarrow \frac{A^2}{2} = \frac{6}{10} \Rightarrow A^2 = \frac{6}{5}$$

$$\Rightarrow \boxed{A = \sqrt{\frac{6}{5}}} \quad \text{So} \quad |\psi(x,0)\rangle = \sqrt{\frac{3}{5}}|\phi_1(x)\rangle + \sqrt{\frac{3}{10}}|\phi_3(x)\rangle + \sqrt{\frac{1}{10}}|\phi_5(x)\rangle$$

(2) $H\phi_1 = E_1\phi_1$ $\qquad E_1, E_2, E_3 = \frac{n^2\hbar^2\pi^2}{2ma^2}$

$H\phi_2 = E_3\phi_3$

$H\phi_5 = E_5\phi_5$

$$E_n = \frac{n^2\hbar^2\pi^2}{2ma^2} = \langle\phi_n|\hat{H}|\phi_n\rangle$$

$$E_1 = \frac{\hbar^2\pi^2}{2ma^2}$$

$$E_3 = \frac{9\pi^2\hbar^2}{2ma^2}$$

$$E_5 = \frac{25\pi^2\hbar^2}{2ma^2}$$

$$\psi(x,0) = \frac{A}{\sqrt{2}}\phi_1(x) + \sqrt{\frac{3}{10}}\phi_3(x) + \sqrt{\frac{1}{10}}\phi_5(x)$$

prob of state $\phi_1(x)$ =

$$P_1(E_1) = |\langle\phi_1|\psi\rangle|^2 = \left|\frac{A}{\sqrt{2}}\right|^2 = \frac{A^2}{2} = \frac{1}{2}\frac{6}{5}$$

$$= \frac{3}{5}$$

$$P_2(E_2) = |\langle\phi_2|\psi\rangle|^2 = \left|\sqrt{\frac{3}{10}}\right|^2 = \frac{3}{10}$$

$$P_3(E_3) = |\langle\phi_3|\psi\rangle|^2 = \left(\sqrt{\frac{1}{10}}\right)^2 = \frac{1}{10}$$

Now Average Energy $= E = \sum_r P_r E_r$

Date:
Page No.:

$$E = P_1 E_1 + P_3 E_3 + P_5 E_5$$

$$E = \frac{3}{5} E_1 + \frac{3}{10} E_3 + \frac{1}{10} E_5 = \left(\frac{6}{10} + \frac{3}{10} \times 9 + \frac{25}{10} \right.$$

$$E = \left(\frac{6 + 27 + 25}{10} \right) \frac{\hbar^2 \pi^2}{2ma^2} = \frac{\cancel{58}^{29}}{10} \frac{\hbar^2 \pi^2}{\cancel{2}ma^2}$$

$$\boxed{E = \frac{29 \pi^2 \hbar^2}{10 ma^2}}$$

(c) Wave funcⁿ at $t = ?$

$$e^{-\frac{i}{\hbar}\hat{H}t} |\psi(x,0)\rangle = e^{-\frac{i}{\hbar}\hat{H}t} \left[\sqrt{\frac{3}{5}} |\phi_1(x)\rangle + \sqrt{\frac{3}{10}} |\phi_3(x)\rangle + \sqrt{\frac{1}{10}} |\phi \right.$$

$$|\psi(x,t)\rangle = \sqrt{\frac{3}{5}} e^{-\frac{i}{\hbar}E_1 t} |\phi_1(x)\rangle + \sqrt{\frac{3}{10}} e^{\frac{-iE_3 t}{\hbar}} |\phi_3(x)\rangle + \sqrt{\frac{1}{10}} e^{\frac{-iE_5 t}{\hbar}} |\phi$$

(d)
$$\phi(x,t) = \sqrt{\frac{2}{a}} \sin\left(\frac{5\pi x}{a}\right) e^{-iE_5 t/\hbar}$$
$$= \phi_5(x) e^{-iE_5 t/\hbar}$$

Prob. of finding the particle in state $\phi(x,t)$ at time t is

$$P = |\langle \phi | \psi \rangle|^2 = \left| \sqrt{\frac{1}{10}} \right|^2 = \frac{1}{10} \quad \text{Ans}$$

Postulates of Q.M. :-

These postulates are the results of experiments.
What is the representation of state & if we know the state then how to find-what will be the state after some time?

(1). State of the system: Acc. to this, the state of any QM system is repⁿ by the state vector $|\phi\rangle$ in Hilbert space. Here ϕ is vector.
In Schⁿ repⁿ, it is represented by $\phi(r)$. Here ϕ is scalar.

Superposition of states (theorem): If $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle \cdots$ $\cdots |\phi_n\rangle$ are orthonormal state vectors then any linear superposition of these state vectors will also be a state vector.

OR

If these vectors repⁿ the state of Q.M. system then any linear superposition of these states will repⁿ also the state of Q.M.

If $\phi_1, \phi_2, \phi_3 \cdots \phi_n$ be the solution of Schⁿ eqⁿ then linear superposition of ~~its~~ these solⁿ will also be the solⁿ of Schⁿ eqⁿ

In Dirac repⁿ, Normalisation condⁿ

Wave funcⁿ; $|\psi\rangle = \sum_n C_n |\phi_n\rangle$

$$\boxed{\langle\psi|\psi\rangle = 1}$$

Hermitian conjugate $\Rightarrow \langle\psi| = \sum_m C_m^* \langle\phi_m|$

In vector, Hermitian conjugate or Complex conjugate is different but in scalar same.

If ψ is normalised to unity then

$$\langle\psi|\psi\rangle = 1$$

Date :
Page No. :

The value of this braket is $\sum_m \sum_n C_m^* C_n \langle \phi_m | \phi$

$$\langle \phi_m | \phi_m \rangle = \delta_{mn}$$
$$= 1, \quad m = n$$
$$= 0 \quad m \neq n$$

$$\sum_m \sum_n C_m^* C_n \langle \phi_m | \phi_n \rangle = 1$$

$$\sum_m \sum_n C_m^* C_n \delta_{mn} = 1$$

$$\boxed{\sum_n |C_n|^2 = 1} \quad (\text{for } n = n)$$

In Sch^n picture,

$$\langle \phi_n | \phi_m \rangle = \int \phi_m^{*} \phi_n \, d\tau = \delta_{mn}$$

(2.) Observable & Operator :- for any obs. or dy-cal variable A there is corresponding Hermition operator $\hat{A}$. And expectation value must be real

$$\hat{A}^+ = A$$
$$\langle \hat{A} \rangle^* = A$$

$$\boxed{\langle \phi_m | \hat{A}^+ | \phi_n \rangle = \langle \phi_m | A | \phi_n \rangle} \text{ — (A)}$$

$$\boxed{(\phi_m, \hat{A} \phi_n) = (\hat{A} \phi_m, \phi_n)} \text{ — (B)}$$

In integral form — if A scalar then complex conjugate of I & II will be same.

$$\boxed{\int \phi_m^+ \hat{A} \phi_n \, d\tau = \int (\hat{A} \phi_m)^+ \phi_n \, d\tau} \text{ — (C)}$$

Any operator should satisfy these cond^n to be a Hermitian operator.

(A), (B), (C) are same.

Energy, position, mom. o/p is Hermitian o/p.

These cond^n are used to calculate the Hermitian o of explicite form as mom. $(-i\hbar \frac{d}{\ }) = i\hbar \frac{d}{\ }$

of O/p are also O/p. $\hat{A}, \hat{A}^2, \hat{A}^3 \ldots$

Date:
Page No.: 65

3.) Measurment & Eigen value of Operators :- The measurment of an o/p observable A may be rep^n by the action of the o/p $\hat{A}$ on the state $|\phi_n\rangle$.

eigen value eq^n

$$\boxed{\hat{A}\,|\phi_n\rangle = \lambda\,|\phi_n'\rangle} \quad (A) \qquad \& \qquad \boxed{\hat{A}\,|\phi_n\rangle = a_n\,|\phi_n\rangle} \quad (B)$$

If an operator, operates on the state & state changes then λ is not the eigen value & if states remain same after operating $\hat{A}$ then a_n is called eigen value & corresponding wave func^n is eigen fuc^n. { In eq^n (A), wave func^n is not eigen fuc^n, it is simply w. func^n }

⇒ The possible results of measurments is given by any one of the eigen values of the operator $\hat{A}$.

→ If we want to calculate the energy eigen value of the operator

Possible ei value of Hamiltonian is given by

$$E_n, \; H = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

4 Probabilistic Outcome of Measurments :- Acc. to this postulate, we can measure the probability of any measurment.

If single state is present then there will be 100 % prob.

When measuring an observable A of any Q. M. system in a state $|\psi\rangle$, the prob. of obtaining one of the non-degenerate eigen value a_n of the corresponding O/p $\hat{A}$ is given by

Date:
Page No.:

$$\boxed{P_n = \frac{|\langle \phi_n | \psi \rangle|^2}{\langle \psi | \psi \rangle}}$$ — (A) → Prob. for -degenerate

& $|\psi\rangle = \sum_n C_n |\phi_n\rangle$

operate an O/p $\hat{A}$,

$\hat{A}|\psi\rangle = \hat{A} \sum_n C_n |\phi_n\rangle$

$= \hat{A}(C_1|\phi_1\rangle + C_2|\phi_2\rangle + \cdots + C_n|\phi_n\rangle)$

$\langle \phi_n | \psi \rangle = C_n$

For Non-degenerate - there will be single state for E. Value.

degenerate - When it may be for degenerate - Suppose t are m states. Calculate the prob. of each state & then average.

The prob. of obtaining m-fold degenerate eigen value a_n of the O/p $\hat{A}$ i given by

$$\boxed{P_n = \frac{\sum_m |\langle \phi_n^m | \psi \rangle|^2}{\langle \psi | \psi \rangle}}$$ → Prob. for degenera case.

If W. fūc^n is superposition of $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle, |\phi_4\rangle$

then $|\psi\rangle = \sum_{n=1}^{4} C_n |\phi_n\rangle$

if $|\phi_1\rangle$ & $|\phi_2\rangle$ are degenerate then

$|\phi_1\rangle \rightarrow |\phi_1^1\rangle$

$|\phi_2\rangle \rightarrow |\phi_1^2\rangle$

i.e. $\phi_n^m \rightarrow$ m different states that have n E. valu

Operator will use to calculate the mom., energy & expectation value but to calculate prob. the O/p is not use.

(5.) Time evalution of state :- Ac...
Know...
t then we can calculate th...
same time.

Suppose at time $t = 0$

$$|\psi(t)\rangle = \hat{U}(t, t_0)\,|\psi(t_0)\rangle$$

$\hat{U}(t, t_0) \rightarrow$ time evalution o/p. We dont know its form. To calculate form of $\hat{U}$, we operate it on following schr eqn,

$$H|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}\psi(t)$$

~~H U~~ Now operate time evalution O/p,

$$H\,\hat{U}(t, t_0)\,|\psi(t_0)\rangle = i\hbar\frac{\partial}{\partial t}\left(\hat{U}(t, t_0)\,|\psi(t_0)\rangle\right)$$

$$= i\hbar\left(\frac{\partial U}{\partial t}\right)|\psi(t_0)\rangle$$

- $|\psi(t_0)\rangle$ is constant bcoz it does not depand on time t.

$$\Rightarrow \boxed{\frac{\partial U}{\partial t} = \frac{-i\hat{H}}{\hbar}\,\hat{U}(t, t_0)}$$

On integrating this eqn, we get $\hat{U}$,

$$\int\frac{\partial U}{U} = \int -\frac{i\hat{H}}{\hbar}\,\partial t + c$$

If H is time independent only then we can integrate it here. so consider H is time independent.

$$\ln\hat{U} = \frac{-i}{\hbar}\hat{H}t + c$$

No 1 :- If $|\psi(t_0)\rangle = \hat{U}(t_0, t_0)\,|\psi(t_0)\rangle$

then $\hat{U} = 1$

Date :
Page No. :

At $t = t_0$, $\hat{U} = 1 \Rightarrow c = \frac{i}{\hbar}\hat{H} t_0$

So. $$\boxed{\hat{U} = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}}$$

This is the form of o/p $\hat{U}$.

By operating this o/p on any state we can calculate the state at time t (after some

- If we know the e value of $\hat{A}$ then we c calculate for $e^{\hat{A}}$.

$$\hat{A}|\phi_n\rangle = a_n|\phi_n\rangle$$

$$e^{\hat{A}} = \left(1 + \hat{A} + \frac{A^2}{2!} + \cdots\right)$$

$$e^{\hat{A}}|\phi_n\rangle = \left(1 + a_n + \frac{a_n^2}{2!} + \cdots\right)|\phi_n\rangle$$

$$= e^{a_n}|\phi_n\rangle$$

(Replace o/p $\hat{A}$ by E. value.)

Prob : Consider a system whose initial state at t= is given in terms of a complete orthonormal set of 3 vectors as $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ as

$$|\psi(x,0)\rangle = \frac{1}{\sqrt{3}}|\phi_1\rangle + A|\phi_2\rangle + \frac{1}{\sqrt{6}}|\phi_3\rangle$$

where A is real const.

(1) find the value of A so that $|\psi(0)\rangle$ is normalised to unity.

(2) If the energies corresponding to $|\phi_1\rangle, |\phi_2\rangle, |\phi$ are given by E_1, E_2 & E_3 respectively. Write do the state of the system at any later tin t.

(3) Determine the prob. of finding the system at time t in the state ϕ_3

(1) $\sum_n |C_n|^2 = 1$

$$|\psi(x,0)\rangle = \frac{1}{\sqrt{3}}|\phi_1\rangle + A|\phi_2\rangle + \frac{1}{\sqrt{6}}|\phi_3\rangle$$

$$\Rightarrow \left(\frac{1}{\sqrt{3}}\right)^2 + A^2 + \left(\frac{1}{\sqrt{6}}\right)^2 = 1$$

$$A^2 + \frac{1}{3} + \frac{1}{6} = 1 \Rightarrow A^2 + \frac{3}{6} = 1$$

$$A^2 = 1 - \frac{1}{2} \Rightarrow A^2 = \frac{1}{2}$$

$$\Rightarrow \boxed{A = \frac{1}{\sqrt{2}}}$$

So $|\psi(x,0)\rangle = \frac{1}{\sqrt{3}}|\phi_1\rangle + \frac{1}{\sqrt{2}}|\phi_2\rangle + \frac{1}{\sqrt{6}}|\phi_3\rangle$

Prob of $|\phi_1\rangle$ state $= |C_1|^2 = \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}$

" $|\phi_2\rangle$ " $= |C_2|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

" $|\phi_3\rangle$ " $= |C_3|^2 = \left(\frac{1}{\sqrt{6}}\right)^2 = \frac{1}{6}$

(2) state of system after some time $t = ?$
So operate time evolution o/p on w. fcn

$$e^{-\frac{i}{\hbar}\hat{H}t}|\psi(x,0)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}\left(\frac{1}{\sqrt{3}}|\phi_1\rangle + A|\phi_2\rangle + \frac{1}{\sqrt{6}}|\phi_3\rangle\right)$$

Replace $\hat{H}$ by Energy Eigen value

$$|\psi(x,t)\rangle = \left(\frac{1}{\sqrt{3}}e^{-i\hat{H}t/\hbar}|\phi_1\rangle + \frac{1}{\sqrt{2}}e^{-i\hat{H}t/\hbar}|\phi_2\rangle + \frac{1}{\sqrt{6}}e^{-i\hat{H}t/\hbar}|\phi_3\rangle\right)$$

$$= \frac{1}{\sqrt{3}}e^{-\frac{iE_1t}{\hbar}}|\phi_1\rangle + \frac{1}{\sqrt{2}}e^{-\frac{iE_2t}{\hbar}}|\phi_2\rangle + \frac{1}{\sqrt{6}}e^{-\frac{iE_3t}{\hbar}}|\phi_3\rangle$$

Date:
Page No.:

Note:- If wave func^n is Normalised at a particular t then it will be always normalised at same time.

(3)

$$|\psi\rangle = \sum_n C_n |\phi_n\rangle$$

$$= C_1|\phi_1\rangle + C_2|\phi_3\rangle + \cdots C_n|\phi_n\rangle$$

Multiply by Hermitian conjugate

$$\langle\phi_n|\psi\rangle = C_1\langle\phi_n|\phi_1\rangle + C_2\langle\phi_n|\phi_2\rangle + \cdots + C_n\langle\phi_n|\phi_n\rangle$$

Prob. of finding the system at t in ϕ_3 state

$$\langle\phi_3|\psi(t)\rangle = \left|\frac{1}{\sqrt{6}} e^{-iE_3t/\hbar}\right|^2$$

$$= \frac{1}{6}\left[e^{-iE_3t/\hbar} \times e^{iE_3t/\hbar}\right]$$

$$= \frac{1}{6}$$

Prob:- A 1 Dim Harmonic oscillator is in the state

$$\psi(x) = 3\psi_0(x) - 2\psi_1(x) + \psi_2(x)$$

where ψ_0, ψ_1 & ψ_2 are the ground, Ist, IInd exited state respectively. The prob. of finding the oscillator in ground state.

$$\text{Prob.} = \frac{|\langle\phi_n|\psi\rangle|^2}{\langle\psi|\psi\rangle}$$

This W.func^n is not normalise i.e:- we have to normalise it so we use above formula.

$$\cancel{P_n = \frac{3}{\sqrt{9+4+1}}\psi_0(x) - \frac{2}{\sqrt{}}}$$

classmate
Date
Page

$\psi(x) = 3\varphi_0(x) - 2\varphi_1(x) + \varphi_2(x)$

Now normalised wave fcn will be

$$\psi(x) = \frac{3\varphi_0(x) - 2\varphi_1(x) + \varphi_2(x)}{\sqrt{9+4+1}}$$

$$\psi(x) = \frac{1}{\sqrt{14}}\left[3\varphi_0(x) - 2\varphi_1(x) + \varphi_2(x)\right]$$

Now, Prob. of finding the particle in ground state,

$$P = |\langle\varphi_0|\psi\rangle|^2$$

$$P = \left|\frac{3}{\sqrt{14}}\right|^2$$

$$\boxed{P = \frac{9}{14}}$$

Que :- The wave fcn of a particle at time $t=0$ is given by $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|u_1\rangle + |u_2\rangle)$

where $|u_1\rangle$ & $|u_2\rangle$ are the normalised eigen stati with eigen values E_1 & E_2 respectively with the cond $(E_2 > E_1)$. The shortest time after which $|\psi(t)\rangle$ will become orthogonal to $|\psi(0)\rangle$ is

(a) $\frac{\hbar\pi}{2(E_2-E_1)}$ (b) $\frac{\hbar\pi}{E_2-E_1}$ (c) $\frac{\sqrt{2}\hbar\pi}{E_2-E_1}$ (d) $\frac{2\hbar\pi}{E_2-E_1}$

To calculate time t, we have to satisfy the condn of orthogonality

$$\langle\psi(t)|\psi(0)\rangle = 0$$

or $$\langle\psi(0)|\psi(t)\rangle = 0$$

Date :

Page No. :

$$e^{-i\hat{H}t/\hbar}\,|\psi(x,0)\rangle = \frac{1}{\sqrt{2}}\left[e^{-\frac{iE_1t}{\hbar}}|u_1\rangle + e^{-\frac{iE_2t}{\hbar}}|\right.$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{iE_1t}{\hbar}}|u_1\rangle + e^{-\frac{iE_2t}{\hbar}}|u_2\rangle\right)$$

or $\langle\psi(t)| = \frac{1}{\sqrt{2}}\left(e^{\frac{iE_1t}{\hbar}}\langle u_1| + e^{\frac{iE_2}{\hbar}}\langle u_2|\right)$ He conju of |

Now, $\langle\psi(t)|\psi(0)\rangle$

$$= \frac{1}{\sqrt{2}}\left(e^{\frac{iE_1t}{\hbar}}\langle u_1| + e^{\frac{iE_2t}{\hbar}}\langle u_2|\right)\left(\frac{1}{\sqrt{2}}(|u_1\rangle + |u\right.$$

$$= \frac{1}{2}\left[e^{\frac{iE_1t}{\hbar}}\langle u_1|u_1\rangle + e^{\frac{iE_2t}{\hbar}}\langle u_2|u_2\rangle\right]$$

$$= \frac{1}{2}\left[e^{\frac{iE_1t}{\hbar}} + e^{\frac{iE_2t}{\hbar}}\right]$$

normalised $\Rightarrow \left\{\begin{array}{l}\langle u_1|u_1\rangle = 1\\ \langle u_2\, u_2\rangle = 1\end{array}\right\}$

<u>Condn of Orthogonality,</u>

$$\langle\psi(t)|\psi(0)\rangle = 0$$

$$\Rightarrow \frac{1}{2}\left(e^{\frac{iE_1t}{\hbar}} + e^{\frac{iE_2t}{\hbar}}\right) = 0$$

$$\Rightarrow e^{\frac{iE_1t}{\hbar}} = -e^{i\frac{E_2t}{\hbar}}$$

$$\Rightarrow e^{iE_2t/\hbar} = \frac{-1}{e^{-iE_1t/\hbar}}$$

$$\Rightarrow e^{i(E_2-E_1)t/\hbar} = -1 = e^{+in\pi}$$ $\left\{\begin{array}{l}\text{Using}\\ \text{DeMoiver's}\\ \text{theorem}\end{array}\right.$

$n = 1, 2, 3, \ldots$

for shortest time $n = 1$

So $e^{i(E_2-E_1)t/\hbar} = e^{+i\pi}$

$$\Rightarrow (E_2-E_1)\frac{t}{\hbar} = \pi$$

$$\boxed{t = \frac{\pi\hbar}{(E_2-E_1)}}$$

(b)

Prob – The component of initial state $|\psi_i\rangle$ of a quantum mechanical system are given in a complete an ortho-normal basis of 3 vectors $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ by.

$$\langle\phi_1|\psi_i\rangle = \frac{2}{\sqrt{3}}$$

$$\langle\phi_2|\psi_i\rangle = \sqrt{\frac{2}{3}}$$

$$\langle\phi_3|\psi_i\rangle = 0$$

Calculate the prob. of finding the system in a state $|\psi_f\rangle$ whose components are given in the same basis by

$$\langle\phi_1|\psi_f\rangle = \frac{1+i}{\sqrt{3}}, \quad \langle\phi_2|\psi_f\rangle = \frac{1}{\sqrt{6}}$$

$$\langle\phi_3|\psi_f\rangle = \frac{1}{\sqrt{6}}$$

$$|\psi_i\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle + c_3|\phi_3\rangle$$

$$|\psi_f\rangle = c_1'|\phi_1\rangle + c_2'|\phi_2\rangle + c_3'|\phi_3\rangle$$

$c_1 = \langle\phi_1|\psi_i\rangle = \frac{2}{\sqrt{3}}$ (To calculate c_1 multiply $|\psi_i\rangle$ by hermitian conjugate of $|\phi_1\rangle$)

$$c_2 = \langle\phi_2|\psi_i\rangle = \sqrt{\frac{2}{3}}$$

$$c_3 = \langle\phi_3|\psi_i\rangle = 0$$

$$c_1' = \langle\phi_1|\psi_f\rangle = \frac{1+i}{\sqrt{3}}$$

$$c_2' = \langle\phi_2|\psi_f\rangle = \frac{1}{\sqrt{6}}$$

$$c_3' = \langle\phi_3|\psi_f\rangle = \frac{1}{\sqrt{6}}$$

Prob. of finding the system in state $|\psi_f\rangle$ is

$$= |c_1'|^2 + |c_2'|^2 + |c_3'|^2$$

$$= \left|\frac{1+i}{\sqrt{3}}\right|^2 + \left|\frac{1}{\sqrt{6}}\right|^2 + \left|\frac{1}{\sqrt{6}}\right|^2$$

Data :
Page No. :

$$= \frac{(1+i)(1-i)}{3} + \frac{1}{6} + \frac{1}{6}$$

$$= \frac{1+1}{3} + \frac{2}{6} \Rightarrow \frac{2}{3} + \frac{2}{\cancel{8}} \Rightarrow \frac{2}{3} + \frac{1}{3}$$

Prob. $= 1$ Ans

Prob :- The wave fun^n of a particle is given by $\psi = \frac{1}{\sqrt{2}}\phi_0 +$ where ϕ_0 & ϕ_1 are the normalised eigen sta with energies E_0 & E_1 corresponding to ground Ist excited state respectively. The expectation v of Hamiltonian in state ψ is

(a) $\frac{E_0}{2} + E_1$ (b) $\frac{E_0}{2} - E_1$ (c) $\frac{E_0 - 2E_1}{3}$ (d) $E_0 +$

$$\psi = \frac{1}{\sqrt{2}}\phi_0 + i\phi_1$$

$$\psi = \frac{1}{\sqrt{2}}\phi_0 + i\phi_1$$

Now, Normalised wave fun^n will be

$$\psi = \frac{\frac{1}{\sqrt{2}}\phi_0 + i\phi_1}{\sqrt{\frac{1}{2} + 1}} = \frac{\frac{1}{\sqrt{2}}\phi_0 + i\phi_1}{\sqrt{\frac{3}{2}}}$$

$$\psi = \frac{1}{\sqrt{3}}\phi_0 + i\sqrt{\frac{2}{3}}\phi_1$$

$$P_0(\phi_0) = |\langle\phi_0|\psi\rangle|^2 = \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}$$

$$P_1(\phi_1) = |\langle\phi_1|\psi\rangle|^2 = \left|\sqrt{\frac{2}{3}}\right|^2 = \frac{2}{3}$$

Average Energy $\langle E\rangle = \sum_r P_r E_r$

$$E = P_0 E_0 + P_1 E_1 = \frac{1}{3}E_0 + \frac{2}{3}E_1$$

$$E = \frac{E_0 + 2E_1}{3}$$

OR

$$\psi = A\left(\frac{1}{\sqrt{2}}\phi_0 + i\phi_1\right)$$

$$\int \psi^* \psi \, d\tau = 1$$

$$\Rightarrow \frac{|A|^2}{2} + |A|^2 = 1 \Rightarrow |A|^2 = \frac{2}{3} \Rightarrow A = \sqrt{\frac{2}{3}}$$

$$\psi = \sqrt{\frac{2}{3}}\left(\frac{1}{\sqrt{2}}\phi_0 + i\phi_1\right) = \frac{1}{\sqrt{3}}\phi_0 + i\sqrt{\frac{2}{3}}\phi_1$$

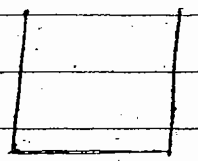
$$\langle E \rangle = P_1 E_1 + P_2 E_2$$

$$= \left|\frac{1}{\sqrt{3}}\right|^2 E_0 + |i|^2 \left|\frac{2}{3}\right|^2 E_1 = \frac{1}{3}E_0 + \frac{2}{3}E_1$$

$$= \frac{E_0 + 2E_1}{3}$$

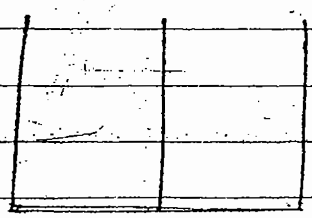
Q4. A particle of mass m in the infinite square well of width 'a' $\left(V(x) = \begin{cases} 0 & 0<x<a \\ \infty & \text{otherwise} \end{cases} \right)$ starts out in the left half of the well & is equally likely to be found in that region. What is the prob. of finding the particle in ground state.

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$



0 a

but Here particle start out in left half of well & is equally likely means prob. is same at every point in well.



0 a/2 a

$$\psi' = A \cancel{\sqrt{\tfrac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)}$$

$$\text{if} \quad \psi'(x) = \begin{cases} \sqrt{\frac{2}{a}} & , 0<x<a/2 \\ 0 & , \text{otherwise} \end{cases}$$

$$\int_0^{a/2} |A|^2 dx = 1$$

$$\Rightarrow A = \sqrt{\frac{2}{a}}$$

Date :
Page No. :

$$\psi'(x) = \sum_n C_n \psi_n = C_1\psi_1 + C_2\psi_2 + \cdots$$

~~To calculate C_1~~

$$C_1 = \langle \psi_1 | \psi' \rangle = \int \psi_1^* \psi' \, dx$$

$$= \int_{-\infty}^{0} 0 \, dx + \int_0^{a/2} \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) \sqrt{\frac{2}{a}} \, dx + \int_{a/2}^{a} \sqrt{\frac{2}{a}}$$

$$+ \int_a^{\infty} 0 \, dx$$

$$= 0 + \int_0^{a/2} \frac{2}{a} \sin\frac{\pi x}{a} \, dx + 0 + 0$$

$$= \frac{2}{a} \left[\cos\frac{\pi x}{a}\right]_0^{a/2} \frac{a}{\pi} = -\frac{2}{\pi}\left[\cos\frac{\pi}{2} - c\right.$$

$$= -\frac{2}{\pi}[0 - 1] = +\frac{2}{\pi}$$

Prob. of finding the particle in ground state.

$$|C_1|^2 = \left|\frac{2}{\pi}\right|^2 = \frac{4}{\pi^2}$$

Ques 1- An e^- is moving freely inside a one Dimensio infinite potn well with walls at $x = 0$ & $x =$ If e^- is initially in the ground state of the box (well) & if we suddenly qudroupole the size of the well (by moving the wall at $x = a$ to $x = 4a$). Calculate the prob. of finding th e^- in

(1) ground state of new box (well)
(2) First Excited state of new box (well)

Initially e^- is in ground state so

$$\psi_1 = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) \quad (n=1)$$

If size changes from a to 4a then wave fu

$$\psi_n' = \sqrt{\frac{2}{4a}} \sin\left(\frac{n\pi x}{4a}\right)$$

$\sin \frac{5\pi}{4} = \sin 225^\circ = \sin(180+45) = \sin 45$

Date:

After changing the size, particle can be in any state.

~~We~~ ∴ Superposition can be done for those w. fcen's which have large no. of states.

Old w. fcen is superposition of new

$$\psi_1 = \sum_n C_1 \psi_n'$$

$$\psi = C_1\psi_1' + C_2\psi_2' + \cdots$$

Prob. P_1 ≠

$$C_1 = \langle \psi_1' | \psi_1 \rangle = \int \psi_1'^* \psi_1 \, dx$$

$$\left\{ \int_{-\infty}^{0} = 0 \quad + \int_0^a \neq 0 \quad + \int_a^{\infty} = 0 \right.$$

$$= \int_0^a \frac{2}{2a} \sin\left(\frac{\pi x}{4a}\right) \sin\left(\frac{\pi x}{a}\right) dx$$

$$= \frac{1}{2}\frac{1}{a}\int_0^a \left[\cos\left(\frac{3\pi x}{4a}\right) - \cos\left(\frac{5\pi x}{4a}\right)\right] dx$$

$$\left\{ \sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right] \right\}$$

$$C_1 = \frac{1}{2a}\left[\frac{4a}{3\pi}\sin\left(\frac{3\pi x}{4a}\right) - \frac{4a}{5\pi}\sin\left(\frac{5\pi x}{4a}\right)\right]_0^a$$

$$= \frac{4a}{2a}\frac{1}{\pi}\left[\frac{1}{3}\sin\left(\frac{3\pi}{4}\right) - \frac{1}{5}\sin\left(\frac{5\pi}{4}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{3}\cos 45^\circ + \frac{1}{5}\sin 45\right] = \frac{2}{\pi}\left[\frac{1}{3}\frac{1}{\sqrt{2}} + \frac{1}{5}\frac{1}{\sqrt{2}}\right]$$

$$= \frac{2}{\pi}\frac{1}{\sqrt{2}}\left[\frac{1}{3} + \frac{1}{5}\right] = \frac{\sqrt{2}}{\pi}\left[\frac{5+3}{15}\right] = \frac{\sqrt{2}}{\pi}\frac{8}{15}$$

$$= \frac{8\sqrt{2}}{15\pi}$$

Probability $P_1 = |C_1|^2 = \frac{(8)^2 \times 2}{(15)^2 \pi^2} = \frac{128}{225 \times 9.8696}$

$$P_1 = \frac{128}{2220.66} = 0.05768 \simeq 0.058$$

$$= 5.8\%$$

Now, $C_2 = \langle \psi_2' | \psi_1 \rangle = \int \psi_2'^* \psi_1 \, dx$

Date :
Page No. :

$$C_2 = \int_0^a \frac{1}{a} \sin\left(\frac{\pi x}{2a}\right) \sin\left(\frac{\pi x}{a}\right) dx$$

$$= \frac{1}{a}\int_0^a \frac{1}{2}\left[\cos\left(\frac{\pi x}{2a}\right) - \cos\left(\frac{3\pi x}{2a}\right)\right] dx$$

$$C_2 = \frac{1}{2a}\left[\frac{2a}{\pi}\sin\left(\frac{\pi x}{2a}\right) - \frac{2a}{3\pi}\sin\left(\frac{3\pi x}{2a}\right)\right]_0^a$$

$$= \frac{2a}{2a}\frac{1}{\pi}\left[\sin\frac{\pi}{2} - \frac{1}{3}\sin\left(\frac{3\pi}{2}\right)\right]$$

$$\cancel{= \frac{1}{\pi}\left[1 - \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)\right] = \frac{1}{\pi}\left[1 - \frac{1}{2\sqrt{3}}\right]}$$

$$= \frac{1}{\pi}\left[\sin 90 - \frac{1}{3}\left[-\sin 90\right]\right] = \frac{1}{\pi}\left[1 + \frac{1}{3}\right]$$

$$= \frac{1}{\pi}\frac{4}{3}$$

$$P_2 = |C_2|^2 = \frac{16}{9\pi^2} = \frac{16}{9\times 9.8696} = \frac{16}{88.8264}$$

$$= 0.18013 \simeq \underline{0.18} = 18\ \% \ \underline{\underline{Ans}}$$

21 Sep 2012

Prob :-

A particle in a deep square well potential extends from $x=0$ to $x=L$ as a wave fuc^n

$$\psi(x) = \frac{1}{\sqrt{5}}\left[|\phi_1\rangle + 2|\phi_2\rangle\right]$$

where $|\phi_1\rangle$ & $|\phi_2\rangle$ denote the ground state & first excited state fuc^n. The expectation value of $\langle x\rangle$ in this state is

(a) $\frac{L}{2}$ (b) $\frac{L}{2} - \frac{48L}{9\pi^2}$

✓(c) $\frac{L}{2} - \frac{64L}{45\pi^2}$ (d) $\frac{L}{2} - \frac{48L}{45\pi^2}$

$$|\psi\rangle = \frac{1}{\sqrt{5}}\left[|\phi_1\rangle + 2|\phi_2\rangle\right] \Rightarrow \langle\psi| = \frac{1}{\sqrt{5}}\left[\langle\phi_1| + 2\langle\phi_2|\right.$$

$$\langle x\rangle = \frac{\langle\psi|x|\psi\rangle}{\langle\psi|\psi\rangle}$$

$\langle\psi|x|\psi\rangle$

$$\langle x\rangle = \int \psi^* x \psi \, dx$$

$$= \frac{1}{5}\left[\langle\phi_1|x|\phi_1\rangle + 2\langle\phi_1|x|\phi_2\rangle + 2\langle\phi_2|x|\phi_1\rangle + 4\langle\phi_2|x|\phi_2\rangle\right]$$

$$\langle\phi_1|x|\phi_1\rangle = \langle\phi_2|x|\phi_2\rangle = L/2$$

& $\langle\phi_1|x|\phi_2\rangle$ & $\langle\phi_2|x|\phi_1\rangle$ are the hermitian conjugat of each other.

$$\langle\phi_1|x|\phi_2\rangle = \int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right) x \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right) dx$$

$$= \frac{1}{L}\int_0^L 2\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right) x\, dx$$

$$\left\{ 2\sin A\sin B = \cos(A-B) - \cos(A+B) \right\}$$

$$\langle\phi_1|x|\phi_2\rangle = \frac{1}{L}\int_0^L\left[\cos\left(\frac{\pi x}{L} - \frac{2\pi x}{L}\right) - \cos\left(\frac{3\pi x}{L}\right)\right] x\, dx$$

$$= \frac{1}{L}\int_0^L\left[\cos\frac{\pi x}{L} - \cos\frac{3\pi x}{L}\right] x\, dx$$

$$= \frac{1}{L}\left[\left(\frac{x\sin\frac{\pi x}{L}}{\frac{\pi}{L}}\right)_0^L + \left(\frac{\cos\frac{\pi x}{L}}{\left(\frac{\pi}{L}\right)^2}\right)_0^L - \left\{\left(\frac{x\sin\frac{3\pi x}{L}}{\frac{3\pi}{L}}\right)_0^L + \left(\frac{\cos\frac{3\pi x}{L}}{\left(\frac{3\pi}{L}\right)^2}\right)_0^L\right\}\right]$$

$$= \frac{1}{L}\left[-\frac{L^2}{\pi^2} - \frac{L^2}{\pi^2} + \frac{L^2}{9\pi^2} + \frac{L^2}{9\pi^2}\right] = \frac{1}{L}\left[-\frac{2L^2}{\pi^2} + \frac{2L^2}{9\pi^2}\right]$$

$$\langle\phi_1|x|\phi_2\rangle = \frac{-16L}{9\pi^2}$$

||ly $\langle\phi_2|x|\phi_1\rangle = \dfrac{-16L}{9\pi^2}$

& $$\langle\phi_1|x|\phi_1\rangle = \frac{2}{L}\int_0^L x\sin^2\frac{\pi x}{L}dx = \frac{1}{L}\int_0^L x\left(1-\cos\frac{2\pi x}{L}\right)dx$$

$$= \frac{1}{L}\left[\frac{x^2}{2} - \frac{\sin\frac{2\pi x}{L}}{2\pi/L}\right]_0^L = \frac{L}{2}$$

$$\langle\phi_2|x|\phi_2\rangle = \frac{L}{2}$$

$$\langle x\rangle = \frac{1}{5}\left[\frac{L}{2} - 2\times\frac{16L}{9\pi^2} - 2\times\frac{16L}{9\pi^2} + \frac{4L}{2}\right] = \frac{L}{2} - \frac{64L}{45\pi^2}$$

Date :
Page No. :

Prob Consider a particle in a 1 Dim potential that sa the cond^n $\boxed{V(x) = V(-x)}$ i.e. pot^n is symmetric. Let $|\psi_0\rangle$ & $|\psi_1\rangle$ denote the ground & Ist exc state respectively. Let $\boxed{|\psi\rangle = \alpha_0|\psi_0\rangle + \alpha_1|\psi}$ be a normalised state with α_0 & α_1 being real constants. The expectation value of x in this s $\{\psi\}$ is given by

(a) $\alpha_0^2\langle\psi_0|x|\psi_0\rangle + \alpha_1^2\langle\psi_1|x|\psi_1\rangle$

(b) $\alpha_0\alpha_1\left[\langle\psi_0|x|\psi_1\rangle + \langle\psi_1|x|\psi_0\rangle\right]$

(c) $\alpha_0^2 + \alpha_1^2$ (d) $2\alpha_0\alpha_1$

$$|\psi\rangle = \alpha_0|\psi_0\rangle + \alpha_1|\psi_1\rangle$$

$$\langle\psi| = \langle\psi_0|\alpha_0 + \langle\psi_1|\alpha_1$$

$$\langle x\rangle = \langle\psi|x|\psi\rangle$$
$$= \langle\psi_0|\alpha_0^2 x|\psi_0\rangle + \langle\psi_0|\alpha_0\alpha_1 x|\psi_1\rangle + \langle\psi_1|\alpha_1\alpha_0 x$$
$$+ \langle\psi_1|\alpha_1^2 x|\psi_1\rangle$$

Now, calculate, $\langle\psi_0|x|\psi_0\rangle$, $\langle\psi_1|x|\psi\rangle$

$\langle\psi_0|x|\psi_1\rangle$, $\langle\psi_1|x|\psi_0\rangle$

If pot^n is symmetric $V(x) = V(-x)$ then hamiltonian will be symmetric

$$H = \frac{p^2}{2m} + V(x)$$

& Hamiltonian commutes with parity.

$$[H, P] = 0$$

If two operator commute, then they have simultaneous eigen func^n.

E func^n of Parity operator have definate pa i.e. E func^n will be even or odd or

" " symmetric or antisymmetric.

$$\langle \psi_0 | x | \psi_0 \rangle = \int_{-\infty}^{+\infty} \psi_0^* \, x \, \psi_0 \, dx$$

(when limit → equal & opposite & integrant → odd then integral → 0)

$\underbrace{|\psi_0|^2}_{\text{even}} \underbrace{x}_{\text{odd}} \Rightarrow$ odd

if pot^n is symmetric then $|\psi_0|^2 \to$ even

So $\langle \psi_0 | x | \psi_0 \rangle = \int_{-\infty}^{+\infty} \psi_0^* \, x \, \psi_0 \, dx = 0$

IIly, $\langle \psi_1 | x | \psi_1 \rangle = \int_{-\infty}^{+\infty} \psi_1^* \, x \, \psi_1 \, dx = 0$ } Put in (1).

So option (b) will be correct becoz result will be in terms of $\langle \psi_0 | x | \psi_1 \rangle$ & $\langle \psi_1 | x | \psi_0 \rangle$ as $\langle \psi_0 | x | \psi_0 \rangle$ & $\langle \psi_1 | x | \psi_1 \rangle$ are zero.

$$\langle x \rangle = \alpha_0 \alpha_1 \left[\langle \psi_0 | x | \psi_1 \rangle + \langle \psi_1 | x | \psi_0 \rangle \right] \quad \text{Ans}$$

Prob :- Consider a particle of mass m subjected to an attractive pot^n δ (dirac-delta)

$$V(x) = -V_0 \, \delta(x) \quad , V_0 > 0$$

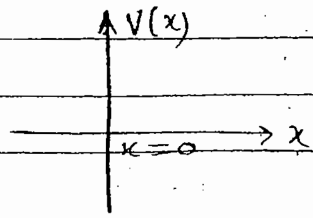
(i) In the case of −ve energies show that this particle has only one bound state. Find the binding energy & corresponding wave func^n.

(ii) In the case of +ve energies. Calculate the reflection & transmission cofficients.

(iii) What is the probability that the particle remains bound when V_0 is (a) halved suddenly,
(b) quadrupole "

{ when pot^n is finite, wave func^n & derivative both continous but if pot^n is infinite then w.func^n is continous but derivative not.

$$\delta x = 0 \quad ; x \neq 0$$
$$= \infty \quad ; x = 0$$



Date :
Page No. :

(i) Particle energy → −ve i.e. $E<0$

In case $\boxed{\begin{array}{l} E<V \Rightarrow \text{bound state} \\ E>V \Rightarrow \text{Unbound} \end{array}}$

If $E>0$ then $E>V$ $(V=-V_0)$ so there will be unbound state not bound so Energy is −ve. (scattering state)

Sch^n eq^n

$$H\psi = E\psi$$

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - V_0\,\delta(x)\right]\psi(x) = E\,\psi(x)$$

If $x \neq 0$, then $\delta(x)=0$ so pot^n $=0$ then

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x)$$

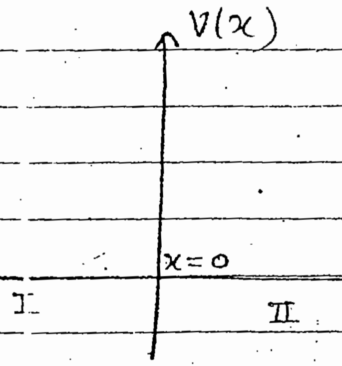
{ Pot^n → Time ind
{ Sch^n eq^n → "

$$\Rightarrow \frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0$$

where, $K^2 = -\frac{2mE}{\hbar^2}$

{ here E → −ve so K ... but K should be +ve take −ve sign }

$$\Rightarrow \frac{d^2\psi}{dx^2} - K^2\psi = 0$$



Solution, $\psi = Ae^{Kx} + Be^{-Kx}$

Wave fun^n should be continous at boundries.

In II Region, $x \to 0$ to $+\infty$

So $Ae^{Kx} \to \infty$ i.e. $A=0$ for II region

So $\boxed{\psi_{II} = Be^{-Kx}}$ —— (A)

In I Region, $x \to -\infty$ to 0

so $Be^{-Kx} \to \infty$ i.e. $B=0$

$\boxed{\psi_I = Ae^{Kx}}$ —— (B)

At boundary i.e. $x=0$, $\psi_I = \psi_{II}$

$$\Rightarrow \boxed{A = B}$$

Pot^n $=\infty$ at boundry & at ∞ pot^n derivative is discontinou then Sch^r eq^n

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\,\delta(x)\,\psi(x) = E\,\psi(x)$$

Here $\frac{d\psi_I}{dx} \neq \frac{d\psi_{II}}{dx}$ so $\frac{d\psi_I}{dx} - \frac{d\psi_{II}}{dx} = ?$

can be calculated from above sch^r eq^n i.e. Amount of discontinuity.

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon}\left(-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\,\delta(x)\,\psi(x)\right)dx = \lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon} E\,\psi(x)\,dx$$

$$\lim_{\epsilon\to 0}\ -\frac{\hbar^2}{2m}\frac{d\psi}{dx}\Big|_{x=-\epsilon}^{x=+\epsilon} - V_0\int_{-\epsilon}^{+\epsilon}\delta(x)\,\psi(x)\,dx = \lim_{\varepsilon\to 0} E\int_{-\epsilon}^{+\varepsilon}\psi(x)\,dx$$

R.H.S. Integral $=0$ bcoz $\psi(x)$ is continous fun^n

i.e. $\int_{-\epsilon}^{+\epsilon}\psi(x)\,dx = 0$.

& $\int_{x_1}^{x_2} f(x)\,\delta(x-a)\,dx = f(a)$ if limit includes
$= 0$ otherwise.

$$\Rightarrow \lim_{\varepsilon\to 0}\ -\frac{\hbar^2}{2m}\left[\frac{d\psi_{II}(x)}{dx}\Big|_{x=+\epsilon} - \frac{d\psi_I}{dx}\Big|_{x=-\epsilon}\right] - V_0\,\psi(0) = 0$$

(B) $\Rightarrow$ $\frac{d\psi_I}{dx} = KAe^{Kx}$ $\Rightarrow$ $\frac{d\psi_I}{dx}\Big|_{x=-\varepsilon} = KAe^{-K\varepsilon} = KA$ $(\varepsilon\to 0)$

(A) $\Rightarrow$ $\frac{d\psi_{II}}{dx} = -KAe^{-Kx}$ $=$ $\frac{d\psi_{II}}{dx}\Big|_{x=\varepsilon} = -KAe^{-K\varepsilon} = -KA$

So $-\frac{\hbar^2}{2m}\left[-KA - KA\right] = V_0 A$

$$\Rightarrow +\frac{\hbar^2}{2m}(2KA) = V_0 A$$

$$\Rightarrow \boxed{K = \frac{mV_0}{\hbar^2}}$$

Date :
Page No. :

There is only single value of K ($m, \hbar, V_0 \to$ contin
So "will be" " " " energy i.e. there i
single bound state.

$$K^2 = -\frac{2mE}{\hbar^2}$$

$$\Rightarrow E = -\frac{\hbar^2 K^2}{2m} \Rightarrow E = \frac{-\hbar^2}{2m}\frac{m^2 V_0^2}{\hbar^4}$$

$$\boxed{E = \frac{-mV_0^2}{2\hbar^2}} \longrightarrow \text{single Value}$$

ψ_I & ψ_{II} can be generalise as $\boxed{\psi = Ae^{-K|x|}}$

or $\boxed{\psi = Ae^{-\frac{mV_0}{\hbar^2}|x|}}$

By Normalization condⁿ, $A = ?$

$$\int_{-\infty}^{+\infty} |\psi|^2 dx = 1$$

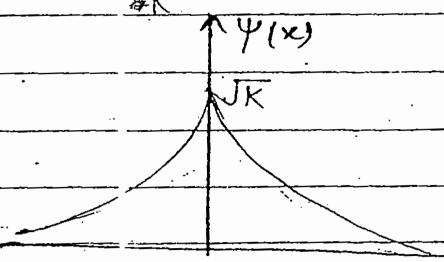
$$|A|^2 \int_{-\infty}^{+\infty} e^{-2K|x|} dx = 1 \Rightarrow 2|A|^2 \int_0^{\infty} e^{-2K|x|} d$$

$$\cancel{2}|A|^2 \left[\frac{e^{-2K|x|}}{-\cancel{2}K}\right]_{0}^{\infty} = 1 \Rightarrow \frac{|A|^2}{-K}[0-1] = 1$$

$$|A|^2 = K$$

$$\Rightarrow \boxed{A = \sqrt{K}}$$

$$\therefore \boxed{\psi = \sqrt{K} e^{-K|x|}}$$



Prob :- A particle in 1 Dim potⁿ has the wave funcⁿ $\psi(x) = \frac{1}{\sqrt{a}} \exp\left[\frac{-|x|}{a}\right]$ where a is a const.

It follows that, for a +ve const. V_0 the potⁿ $V(x)$ is equal to

(a) $V_0 x^2$ (b) $V_0|x|$ ✓(c) $-V_0 \delta(x)$ (d) $\frac{-V_0}{|x|}$

This w-funⁿ. will have discontinous derivative at $x=0$
(from previous prob.)

We know that, when potⁿ $\to \infty$ then I derivative of w-funⁿ will be discontinous.

So at $x=0$, potⁿ includes dirac-delta funⁿ.
i.e. potⁿ will be of dirac-delta type.

(c) is correct.

(ii) +ve energy & R, T = ?

Schʳ eqⁿ,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} - V_0\delta(x)\psi(x) = E\psi(x)$$

$x \neq 0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi(x)$$

$$\Rightarrow \frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0$$

Here $E \to$ +ve & K should be +ve also so

take $$K^2 = \frac{2mE}{\hbar^2}$$

$$\therefore \frac{d^2\psi}{dx^2} + K^2\psi = 0$$

Solⁿ, $$\psi = Ae^{iKx} + Be^{-iKx}$$

In I region $\to$ there is chance of reflection (more)
& also " " transmission

II " $\to$ No reflection only transmission (bcoz No boundry)

$$\psi_I = Ae^{ikx} + Be^{-ikx} \quad \& \quad \psi_{II} = Ce^{ikx}$$

Here no bound state so can not calculate energy only Reflection & transmission coffi. can be calculated.

$$R = \left|\frac{S_R}{S_I}\right| \qquad T = \left|\frac{S_T}{S_I}\right|$$

Date :
Page No. :

Incident wave $\psi_I = Ae^{ikx}$

Reflected wave $\psi_R = Be^{-ikx}$

Transmitted wave $\psi_T = ce^{ikx}$

We have Current density,

$$S = \frac{-i\hbar}{2m}\left[\psi^* \nabla\psi - \psi\nabla\psi\right]$$

$$S_I = \frac{-i\hbar}{2m}\left[A^2 e^{-ikx} \cdot ike^{ikx} + A^2 e^{ikx} ike^{ikx}\right]$$

$$= \frac{-i\hbar}{2m} A^2 [ik + ik] = \frac{-i\hbar}{2m} A^2 \cdot 2ik$$

$$S_I = \frac{\hbar K}{m} A^2$$

$$S_R = \frac{-i\hbar}{2m}\left[B^2 e^{ikx}(-ik)e^{-ikx} - B^2 e^{-ikx}(ik)e^{ikx}\right]$$

$$= \frac{-i\hbar}{2m} B^2 [-ik - ik] = \frac{-i\hbar}{2m} B^2(-2ik)$$

$$S_R = \frac{-\hbar K}{m} B^2$$

$$S_T = \frac{-i\hbar}{2m}\left[c^2 ik + c^2 ik\right]$$

$$= \frac{-i\hbar}{2m} c^2 2ik$$

$$S_T = \frac{\hbar K}{m} c^2$$

So Reflection Coefficient $R = \left|\frac{S_R}{S_I}\right| = -\frac{B^2}{A^2}$

$$\boxed{R = \left|\frac{S_R}{S_I}\right| = \left|\frac{B}{A}\right|^2}$$

Transmission Coefficient $\boxed{T = \left|\frac{S_T}{S_I}\right| = \left|\frac{C}{A}\right|^2}$

(iii)

$$\psi(x = \sqrt{K}\, e^{-K|x|}$$

$$K = \frac{mV_0}{\hbar^2}$$

$$\therefore\ \psi(x) = \sqrt{\frac{mV_0}{\hbar^2}}\cdot e^{-\frac{mV_0}{\hbar^2}|x|}$$

new w.f., $\psi'(x) = \sqrt{\frac{mV_0'}{\hbar^2}}\, e^{-\frac{mV_0'}{\hbar^2}|x|}$ $(V_0 \to V_0')$

$$\cancel{\psi'(x) = \sum_n c_n \psi_n(x)}$$

$$\psi'(x) = c\,\psi(x)$$

$$\therefore\ \int_{-\infty}^{+\infty} \psi^*(x)\,\psi'(x)\,dx = c$$

$$\&\quad P = |c|^2$$

$$\int_{-\infty}^{\infty} \sqrt{\frac{mV_0}{\hbar^2}}\sqrt{\frac{mV_0'}{\hbar^2}}\, e^{-\frac{mV_0}{\hbar^2}|x|}\, e^{-\frac{mV_0'}{\hbar^2}|x|}\,dx = c$$

$$2\frac{m}{\hbar^2}\int_0^{\infty} \sqrt{V_0V_0'}\, e^{-\frac{m}{\hbar^2}|x|(V_0+V_0')}\,dx = c$$

$$2\frac{m}{\hbar^2}\sqrt{V_0V_0'}\left[\frac{e^{-\frac{m}{\hbar^2}(V_0+V_0')|x|}}{-\frac{m}{\hbar^2}(V_0+V_0')}\right]_0^{+\infty} = c$$

$$\frac{2\frac{m}{\hbar^2}\sqrt{V_0V_0'}}{-\frac{m}{\hbar^2}(V_0+V_0')}\left[0-1\right] = c \qquad \left\{\begin{matrix}(e^{-\infty}-e^{-0})\\(0-1)\end{matrix}\right.$$

$$c = \frac{2\sqrt{V_0V_0'}}{(V_0+V_0')}$$

$$P = |c|^2 = \frac{4V_0V_0'}{(V_0+V_0')^2}$$

(a) $V_0 = \frac{V_0'}{2}$

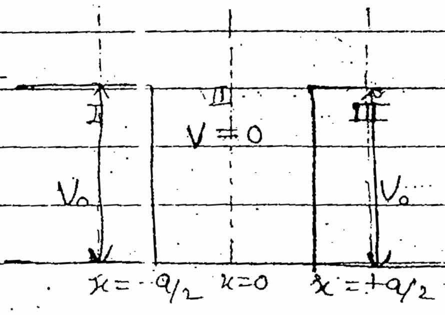
$$P = \frac{4V_0\frac{V_0}{2}}{(\frac{3}{2}V_0)^2} = \frac{8}{9}$$

(b) $V_0 = \dfrac{V_0'}{4}$

$$P = \frac{V_0^2}{\left(\frac{5}{4}\right)^2 V_0^2} = \frac{16}{25} \quad \text{Ans}$$

Finite Potential Well :-

$$V(x) = \begin{cases} 0, & |x| < a/2 \\ V_0, & |x| > a/2 \end{cases}$$



for I & III region,
Sch^r eq^n,

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}[E - V_0]\psi(x) = 0$$

In II Region,

$$\frac{d^2\psi(x)}{dx^2} + \frac{2mE}{\hbar^2}\psi(x) = 0$$

$$K_I^2 = K_{III}^2 = \frac{2m(V_0 - E)}{\hbar^2}$$

$$K_{II}^2 = \frac{2mE}{\hbar^2}$$

So $$\frac{d^2\psi_I}{dx^2} - K_I^2\psi_I(x) = 0$$

$$\frac{d^2\psi_{III}}{dx^2} - K_{III}^2\psi_{III}(x) = 0$$

$$\frac{d^2\psi_{II}}{dx^2} + K_{II}^2\psi_{II}(x) = 0$$

Sol^n,

$$\psi_I = Ae^{K_1 x} + Be^{-K_1 x}$$

$$\psi_{III} = Ee^{+K_1 x} + Fe^{-K_1 x}$$

$$\psi_{II} = C\sin K_{II}x + D\cos K_{II}x$$

In I Region, At $x = -\infty$, $B = 0$

In III region, At $x = \infty$ so $E = 0$

$$\boxed{\psi_{III} = F e^{-K_I x}}$$

In II region, Finite region so x cant be $+\infty$ or $-\infty$

So $$\boxed{\psi_{II} = C \sin K_{II} x + D \cos K_{II} x}$$

At boundry (finite), W. fucⁿ & its derivative is continous.

In ψ_{II} → $\sin$ → odd fucⁿ as, $\sin(-x) = -\sin x$
$\cos$ → Even fucⁿ as, $\cos(-x) = \cos x$

(i) Even fucⁿ

$$\psi_I = A e^{K_I x}$$

$$\psi_{II} = D \cos K_{II} x$$

$$\psi_{III} = F e^{-K_I x}$$

B.C., $\psi_I \Big|_{x=-a/2} = \psi_{II}\Big|_{x=-a/2}$ | $\psi_{II}\Big|_{x=a/2} = \psi_{III}\Big|_{x=a/2}$

$\dfrac{d\psi_I}{dx}\Big|_{x=-a/2} = \dfrac{d\psi_{II}}{dx}\Big|_{x=-a/2}$ | $\dfrac{d\psi_{II}}{dx}\Big|_{x=a/2} = \dfrac{d\psi_{III}}{dx}\Big|_{x=a/2}$

$$\Rightarrow A e^{K_I\left(-\frac{a}{2}\right)} = D \cos K_{II}\left(-\frac{a}{2}\right) \qquad F e^{-K_I a/2} = D \cos K_{II}\, a/2 \quad \text{—(2)}$$

$$A e^{-K_I a/2} = D \cos K_{II}\, a/2 \quad \text{—(1)}$$

$$\Rightarrow \boxed{A = F}$$

Now derivative B.C. ⇒ $K_I A e^{-K_I a/2} = -D K_{II} \sin K_{II}(-a/2)$
$= D K_{II} \sin K_{II}\, a/2$ —(3)

& $K_I F e^{-K_I a/2} = D K_{II} \sin K_{II}\, a/2$ —(4)

eqⁿ (3)/(2) → $\boxed{K_I = K_{II} \tan K_{II} \frac{a}{2}}$ ← (5)

We have $K_I^2 = \dfrac{2m(V_0 - E)}{\hbar^2}$ & $K_{II}^2 = \dfrac{2mE}{\hbar^2}$

Now $K_I^2 + K_{II}^2 = \dfrac{2mV_0}{\hbar^2}$ —(6)

$\eta^2 + \xi^2 = R^2 = \frac{2m}{\hbar^2} V_0 a^2$ ✓

Date :

Page No. :

Put the value of K_I then find K_{II} & then k

$(6) \Rightarrow \left(\frac{a}{2}\right)^2 [K_I^2 + K_{II}^2] = \frac{2 m V_0}{\hbar^2} \frac{a^2}{4}$

$(6) \Rightarrow \left(\frac{a}{2}\right)^2 [K_I^2 + K_{II}^2] = \frac{m V_0}{\hbar^2} \frac{a^2}{2}$

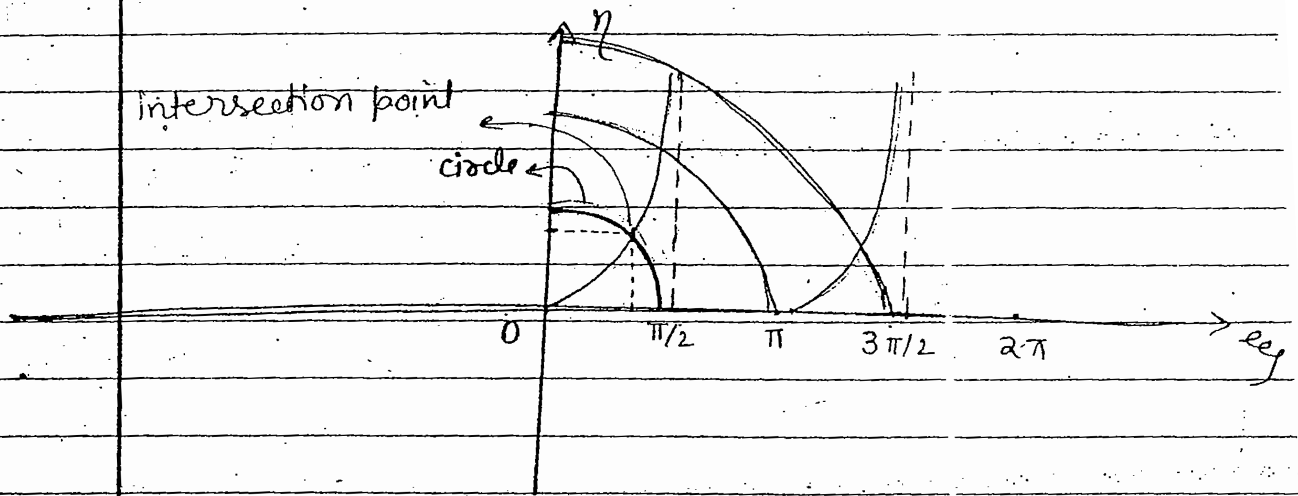
Here we can find K_I w.r.t. to K_{II} & K_{II} w.r.t. So in case of finite potⁿ well we'll get energy states.

Assume, $\xi = K_{II} \frac{a}{2}$ & $\eta = K_I \frac{a}{2}$

$(5) \Rightarrow \frac{a}{2} K_I = \frac{a}{2} K_{II} \tan K_{II} \frac{a}{2}$

$$\Rightarrow \boxed{\eta = \xi \tan \xi}$$

& $$\boxed{\eta^2 + \xi^2 = \frac{m V_0 a^2}{2\hbar^2} = R^2} = R^2$$ Repⁿ the circ $a^2 = $ (radiu



$-$ve values of η are not allowed

$\frac{\pi}{2} \to \pi \Rightarrow \tan(-ve)$ so no curve in this limi

$\frac{3\pi}{2} \to 2\pi \Rightarrow$ " " "

~~$R < \frac{\pi}{2}$~~ If $0 < R < \frac{\pi}{2} \Rightarrow$ one bound state as 1 pt. of intersec

$0 < R < \pi \Rightarrow$ one bound stat

Date :
Page No. :

$0 < R < \frac{3\pi}{2} \Rightarrow$ Bound state may be 1 or 2

$\pi < R < \frac{3\pi}{2} \Rightarrow$ 2 intersection pt. So 2 bound state.

$R \uparrow$, bound state $\uparrow$ bcoz $V_0 a^2 \uparrow$.

(ii) For Odd fun^n

$$\eta = -\xi \cot \xi$$
$$= \cancel{\xi \tan\left(\frac{\pi}{2} + \xi\right)}$$

This plot will shift by $\pi/2$. & in this case the region not allowed in even case will be allowed.

So we'll get alternative regions allowed in even & odd region.

for complete plot, we combine the even & odd plot.

$$\psi_{I} = A e^{K_I x}$$
$$\psi_{II} = C \sin K_{II} x$$
$$\psi_{III} = F e^{-K_I x}$$

On applying B.C., we get

~~A = B~~

& $K_I = K_{II} \cot K_{II} \frac{a}{2}$ —— (i)

& also we have $K_I^2 + K_{II}^2 = \frac{2mV_0}{\hbar^2}$ —— (ii)

multiply by $a/2$ in (i) & $(a/2)^2$ in (ii), we get

$$\frac{a}{2} K_I = -\frac{a}{2} K_{II} \cot K_{II} \frac{a}{2}$$

$$\left(\frac{a}{2}\right)^2 K_I^2 + K_{II}^2 = \frac{2mV_0}{\hbar^2}\left(\frac{a}{2}\right)^2$$
$$= \frac{mV_0}{\hbar^2} \cdot \frac{a^2}{2}$$

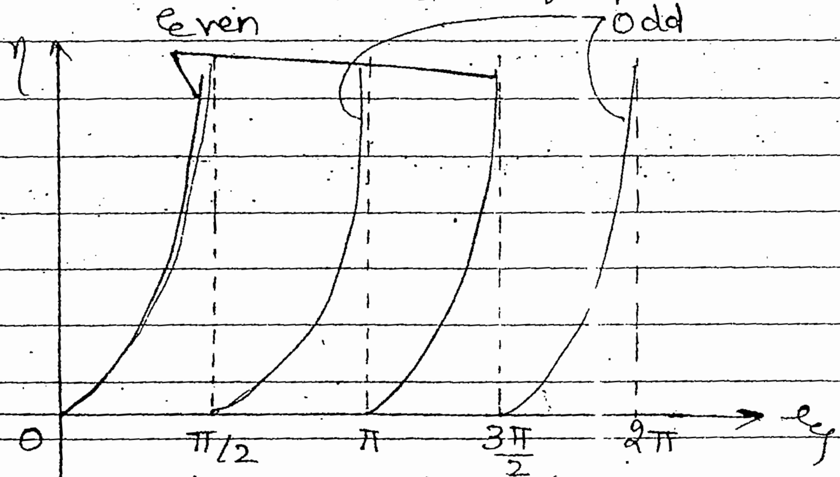
Assume, $\eta = \frac{a}{2} K_I$

& $\xi = \frac{a}{2} K_{II}$

then we get $\boxed{\eta = -\xi \cot \xi = \xi \tan\left(\frac{\pi}{2}\right.}$

& $\boxed{\eta^2 + \xi^2 = \frac{m V_0 a^2}{2\hbar^2} = R^2}$

No. of bound states varies for range of V_0
$V_0 L^2$ is called Strength of potⁿ well.



R	No. of bound State	Even	Odd
$0 < R < \frac{\pi}{2}$	1	1	0
$\frac{\pi}{2} < R < \pi$	2	1	1
$\pi < R < \frac{3\pi}{2}$	3	2	1
$\frac{3\pi}{2} < R < 2\pi$	4	2	2

* ~~If~~ Width of ~~p~~ finite potⁿ well changes :- if width
$-a$ to a then $K_I = K_{II} \tan K_{II} a$

$\Rightarrow a K_I = a K_{II} \tan K_{II} a$ $\left\{ \begin{array}{l} \eta = K_I \\ \xi = K \end{array} \right.$

and $a^2 (K_I^2 + K_{II}^2) = \frac{2 m V_0 a^2}{\hbar^2}$

$[\eta = \xi \tan \xi \; ; \; \eta^2 + \xi^2 = R^2]$ where $R^2 = 2 m V_0 a$

Compare the energy of infinite & finite Potⁿ well :-

$\eta = \xi \tan \xi$

$$\eta^2 + \xi^2 = R^2 = \frac{m V_0 a^2}{4\hbar^2}$$

To exist single bound state $R < \pi/2$

for ground state, $\xi < \frac{\pi}{2}$

$$K_{II} \, a/2 < \frac{\pi}{2}$$

$$\Rightarrow K_{II}^2 \frac{a^2}{4} < \frac{\pi^2}{4} \Rightarrow \frac{2mE}{\hbar^2}\frac{a^2}{4} < \frac{\pi^2}{4}$$

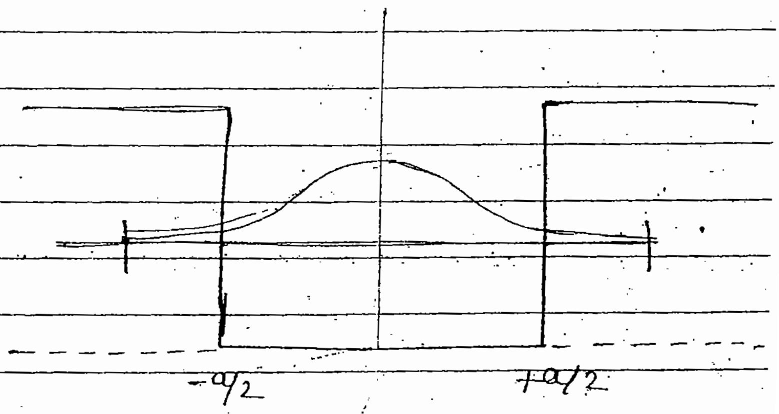
$$\Rightarrow E < \frac{\hbar^2 \pi^2}{2ma^2}$$

In general $\boxed{E_n < \frac{n^2 \pi^2 \hbar^2}{2ma^2}}$

Effective width of potⁿ well increased.

$E = \frac{hc}{\lambda}$

As $\lambda \uparrow$ so $E \downarrow$,



Particle do back & forth motion in the region.

Prob There are only ~~one~~ 3 bound state for a particle of mass m in a 1 Dim potⁿ well of the form shown in figure.

The depth V_0 satisfies the condⁿ,

(a) $\frac{2\pi^2\hbar^2}{ma^2} < V_0 < \frac{9\pi^2\hbar^2}{2ma^2}$

(b) $2\pi^2\hbar^2$ . . . . $\frac{4\pi^2\hbar^2}{}$

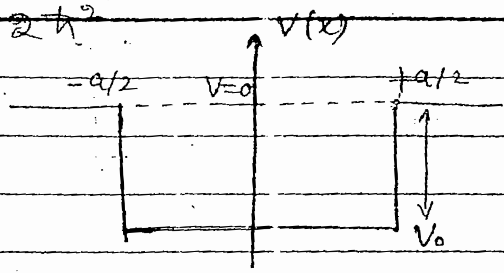
Date :
Page No. :

$$\therefore \eta^2 + \xi^2 = R^2 = \frac{2mV_0}{\hbar^2}\left(\frac{a}{2}\right)^2$$

$$\pi = \frac{mV_0 a^2}{2\hbar^2}$$

(c) $\frac{2\pi^2\hbar^2}{ma^2} < V_0 < \frac{8\pi^2\hbar^2}{ma^2}$

(d) $\frac{2\pi^2\hbar^2}{ma^2} < V_0 < \frac{50\pi^2\hbar^2}{ma^2}$



To exist 3 bound state, radius should be

$\pi < R < 3\pi/2$

& energy should be larger than energy for $n=2$.
" " smaller " " $n=3$

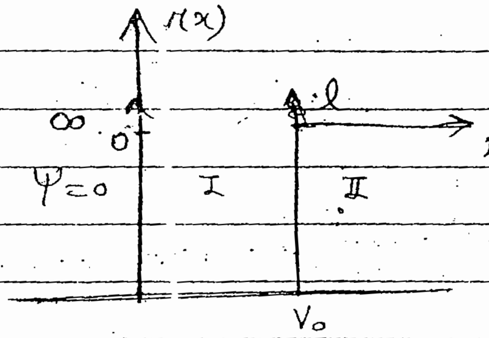
✓(a)

Prob :- A particle in 1 Dim potⁿ

$$V(x) = \begin{cases} \infty & \text{if } x<0 \\ -V_0 & \text{if } 0 \le x \le d \\ 0 & \text{if } x>d \end{cases}$$

If there is at least one bound state, the minimum depth of the potⁿ is

✓(a) $\frac{\hbar^2\pi^2}{8ml^2}$ (b) $\frac{\hbar^2\pi^2}{2ml^2}$ (c) $\frac{2\hbar^2\pi^2}{ml^2}$ (d) $\frac{\hbar^2\pi^2}{ml^2}$



$0 < R < \pi/2$

$\psi_I = 0$

If $E > 0$ then we'll get scattering states so for bound state take $E \to -ve$

i.e. $E < 0$

$$K_2^2 = -\frac{2mE}{\hbar^2}$$

$$K_1^2 = \frac{2m}{\hbar^2}(V_0 + E)$$

$$K_1^2 + K_2^2 = R^2$$

$$\left\{ \begin{array}{l} \cancel{\eta = \xi \tan \xi} \\ \eta = -\xi \cot(K_{II} l) \end{array} \right.$$

multiply (by the coffi. of K_{II})

$K_1^2 + K_2^2 \Rightarrow$
$$-\cancel{\frac{2mE}{\hbar^2}} + \frac{2mV_0}{\hbar^2} + \cancel{\frac{2mE}{\hbar^2}} \cancel{= R^2}$$

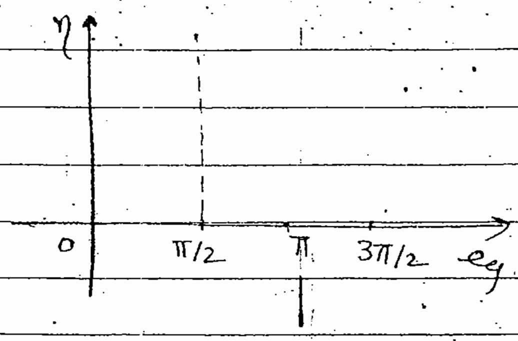
$$K_1^2 + K_2^2 = \frac{2mV_0}{\hbar^2}$$

$$l^2 (K_1^2 + K_2^2) = \frac{2mV_0}{\hbar^2} l^2$$

Now assume $\xi = K_2 l$
$\eta = K_1 l$

$$\boxed{\eta^2 + \xi^2 = \frac{2mV_0}{\hbar^2} l^2 = R^2} \quad \text{——(A)}$$

for one bound state



$$\frac{\pi}{2} < R < \pi$$

Min value of $R = \pi/2$

So $(A) \Rightarrow \dfrac{\pi^2}{4} = \dfrac{2mV_0 l^2}{\hbar^2}$

$$\Rightarrow \boxed{V_0 = \frac{\pi^2 \hbar^2}{8ml^2}}$$

∴ (a) is correct.

22-Sep-2012

Prob :- A particle of mass m is confined to a finite square well from $x = -\frac{L}{2}$ to $x = +\frac{L}{2}$. If the particle is in its ground state and has the wave funcⁿ $\psi = A$ at $x = 0$
& $\psi = \frac{\sqrt{3}}{2} A$ at $x = \frac{L}{4}$

(i) The value of the wavefuncⁿ at $x = \frac{L}{2}$ is

(a) A (b) $\frac{A}{2}$ (c) $\frac{\sqrt{3}}{2} A$ (d) 0

Date :
Page No. :

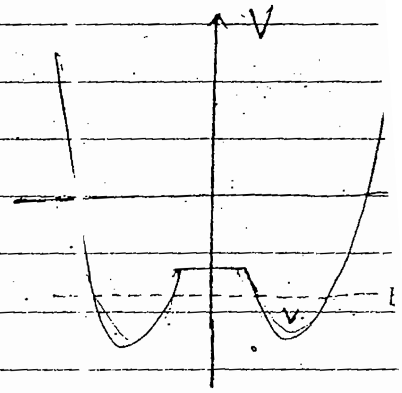
Prob :- A particle with energy E is in a time indepe double well potⁿ as shown in the figure. Which of the following statement about the p is not correct:

(i) The particle be always be in a bound state.

(ii) ~~The probability of finding the particle in one wel~~ will time dependent.

(iii) The particle will be confine to any one of the well.

(iv) The particle can tunnel from one well to other & back.



Particle is not doing free motion so it is in bound state.

If potⁿ is time independent then prob is alwa time independent

(ii)

Prob :- A particle of mass m is in the ground state o the infinite square well.

$$V(x) = \begin{cases} 0 & , \quad 0<x<a \\ \infty & , \quad \text{otherwise} \end{cases}$$

Suddenly the well expands to twice its ori sides (the R.H. wall moving from a to 2a) leaving the wave funcⁿ undisturb. The energy o the particle is measured. The most probable value of energy is

(a) $\frac{\pi^2\hbar^2}{8ma^2}$ (b) $\frac{\pi^2\hbar^2}{2ma^2}$ (c) $\frac{5}{16}\frac{\pi^2\hbar^2}{ma^2}$ (d) 0

$$\phi(x) = \sqrt{\frac{2}{a}} \sin\frac{\pi x}{a}$$

$$\psi(x) = \sqrt{\frac{2}{2a}} \sin\left(\frac{n\pi x}{2a}\right)$$

$$\phi(x) = \sum_n C_n \psi_n x$$
$$= C_1\psi_1 + C_2\psi_2 + C_3\psi_3 \cdots\cdots$$

$$C_n = \int_{-\infty}^{+\infty} \psi_n^* \phi(x)\, dx = \int_0^a \frac{2}{a}\cdot\frac{1}{\sqrt{2}} \sin\left(\frac{n\pi x}{2a}\right)\cdot \sin\left(\frac{\pi x}{a}\right) dx$$

$$= \frac{1}{\sqrt{2}a}\int_0^a 2\sin\left(\frac{n\pi x}{2a}\right)\sin\left(\frac{\pi x}{a}\right) dx$$

$$= \frac{1}{\sqrt{2}a}\int_0^a \left[\cos\frac{(n-2)\pi x}{2a} - \cos\frac{(n+2)\pi x}{2a}\right] dx$$

$$= \frac{1}{\sqrt{2}a}\left[\frac{2a}{(n-2)\pi}\sin\frac{(n-2)\pi x}{2a} - \frac{2a}{(n+2)\pi}\sin\frac{(n+2)\pi x}{2a}\right]_0^a$$

$$= \frac{1}{\sqrt{2}a}\,\frac{2a}{\pi}\left[\frac{1}{(n-2)}\sin\frac{(n-2)\pi}{2a}\cdot a - \frac{1}{(n+2)}\sin\frac{(n+2)\pi}{2a}\cdot a\right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{1}{(n-2)}\sin\frac{(n-2)\pi}{2} - \frac{1}{(n+2)}\sin\frac{(n+2)\pi}{2}\right]$$

$$P = |C_n|^2 = \frac{2}{\pi^2}\left[\frac{1}{(n-2)}\sin\frac{(n-2)\pi}{2} - \frac{1}{(n+2)}\sin\frac{(n+2)\pi}{2}\right]^2$$

$n=1$, $$P_1 = \frac{2}{\pi^2}\left[\frac{1}{(-1)}\sin\left(\frac{-\pi}{2}\right) - \frac{1}{3}\sin\frac{3\pi}{2}\right]^2$$

$$= \frac{2}{\pi^2}\left[1+\frac{1}{3}\right]^2 = \frac{2}{\pi^2}\left(\frac{4}{3}\right)^2 = \frac{16\times 2}{9\pi^2}$$

$$P_1 = \frac{32}{9\pi^2}$$

$n=2$, $$C_2 = \int_0^a \frac{2}{\sqrt{2}a}\sin\left(\frac{2\pi x}{2a}\right)\sin\left(\frac{\pi x}{a}\right) dx$$

$$= \frac{\sqrt{2}}{a}\int_0^a \sin^2\left(\frac{\pi x}{a}\right) dx = \frac{\sqrt{2}}{a}\int_0^a \left(\frac{1-\cos\frac{2x\pi}{a}}{2}\right) dx$$

$$= \frac{\sqrt{2}}{a}\left[\frac{1}{2}x - \left(\frac{\sin\frac{2\pi x}{a}}{4\pi/a}\right)\right]_0^a = \frac{\sqrt{2}}{a}\left[\frac{a}{2}\right]$$

$$= \frac{1}{\sqrt{2}}$$

$$P_2 = |C_2|^2 = \frac{1}{2}$$

Date :
Page No. :

Dirac Notation Property :-

OR

Ket - Bra vectors properties :-

Space vector is independent of representation of but component depend on "

e.g. $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= ()\hat{r} + ()\hat{\theta} + ()\hat{\phi}$

$= ()\hat{s} + ()\hat{\phi} + ()\hat{z}$

In Sch^r rep^n $\Rightarrow$ state is represented by sca
but In Dirac rep^n $\rightarrow$ " " " " vect

In dirac rep^n, unit vectors rep^n basis.
for a vector, there may be ∞ no. of ba
Vector will be basis vector if they satisfy t
orthonormality cond^n i.e.

$\hat{i}\cdot\hat{i} = 1, \quad \hat{i}\cdot\hat{j} = 0, \quad \hat{i}\cdot\hat{k} = 0$

If 2 state vectors $|\phi_1>$ & $|\phi_2>$ then

$\Rightarrow$ If they have some relationship then by chang $\phi_1 \rightarrow, \phi_2$ will be change so the relation rema same.

relation :$\Rightarrow |\phi_1> + 2|\phi_2> = 0$

Then these vectors are Linearly dependent vecto

$\Rightarrow$ But In Linearly Independent $\rightarrow$ No relation b/w vectors state.

$\Rightarrow$ Consider a linear combination of state

$|\phi_1>, |\phi_2 \;----\; |\phi_n$

$\sum_n C_n |\phi_n> = 0$

$\Rightarrow$ If $C_n = 0 \Rightarrow$ Linearly dependent

$C_n \neq 0 \Rightarrow$ Linearly Independent

for Ex $\rightarrow$ If we have 10 vectors & $C_1 = C_3 \neq 0$ &
all other 0 then ~~C~~ vectors corresponding to $C_1, C_3 \rightarrow$ L.I

⇒ Corresponding to this, vector ~~form~~ by its hermitian conjugate (Hermitian Conj. of ~~bra~~ vector → ket vector) will rep^n the Dual Space.

Properties

(In case of vector we take $+$ in place of $*$ (scalar).

1) $\langle\psi| = (|\psi\rangle)^+$

2) $\boxed{|a\psi\rangle = a|\psi\rangle}$

$\langle a\psi| = ?$

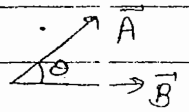
$(|a\psi\rangle)^+ = (a|\psi\rangle)^+$

$\boxed{\langle a\psi| = \langle\psi|a^*}$

3) $(\langle\psi|\psi\rangle)^{1/2} =$ Norm or length of vector

4) $\langle\phi|\psi\rangle =$ projection of 1 vector along the dir^n of other vector.

→ Scalar product of 2-state vectors.



$A\cdot B = |A|\,|B|\cos\theta$ → Projection

(5) $|\psi\rangle = \sum_n C_n|\phi_n\rangle$

Vector Product → If we take cross product of $|\phi\rangle$ $|\psi\rangle$ then

→ If both vectors belong to same vector space then vector product will be forbidden.

→ But if both vectors belong to ~~same~~ different vector space then product will be allowed.

e.g. $|\psi\rangle = |\psi\rangle_{space}, |\psi\rangle_{orb}, |\psi\rangle_{spin}, |\psi\rangle_{isospin}$

Date :
Page No. :

(6) If product of $\langle\psi|$ & $(|C_1\phi_1\rangle + |C_2\phi_2\rangle)$

- $\langle\psi_1|(|C_1\phi_1\rangle + |C_2\phi_2\rangle) = C_1\langle\psi_1|\phi_1\rangle + C_2\langle\psi_1|\phi_2\rangle$
- $(\langle C_1\phi_1| + \langle C_2\phi_2|)|\psi_1\rangle = C_1^*\langle\phi_1|\psi_1\rangle + C_2^*\langle\phi_2|\psi_1\rangle$

<u>Operator</u> :- A operator is a mathematical ru[le] when operator operates on the ket vector, it transform into another ket vec[tor] of the same space.

$$\hat{A}|\psi\rangle = |\psi'\rangle$$

When operator on bra vector, it transform to another bra~~s~~ vector of same space.

$$\langle\psi|\hat{A} = \langle\psi'|$$

$\hat{A}|\psi\rangle = \underbrace{\lambda|\psi\rangle}$ $\{\lambda|\psi\rangle = |\psi'\rangle\}$

~~State of ψ is same then ψ is Eigen func^n~~ but if

$\hat{A}|\psi\rangle = (\lambda)|\psi'\rangle$ then ψ is just <u>wave func^n</u>.

<u>Linear Operator</u> :- An operator is said to be linear if it commute with constant & follow distributive property or dis. law.

$$\hat{A}(C_1|\phi_1\rangle + C_2|\phi_2\rangle) = C_1\hat{A}|\phi_1\rangle + C_2\hat{A}|\phi_2\rangle$$

$\hat{A} \longrightarrow$ Linear operator.

<u>Ex</u> $\rightarrow$ (1) Identity operator
(2) Null "
(3) $|\phi\rangle\langle\psi|$

<u>Note</u> :- $\langle\phi|\psi\rangle \Rightarrow$ This is not operator.
This is <u>Number</u>

Hermitian Operator :- Any operator is said to be Hermitian if it has real expectation value of follow the following condition

$$\langle \hat{A} \rangle^* = \langle \hat{A} \rangle$$

$$(\langle \phi | \hat{A} | \psi \rangle)^\dagger = \langle \psi | \hat{A}^\dagger | \phi \rangle$$
$$= \langle \psi | \hat{A} | \phi \rangle$$

Schⁿ eqⁿ $\Rightarrow (\phi, \hat{A}\psi) = (\hat{A}\phi, \psi)$

$$\Rightarrow \int \phi^* \hat{A} \psi \, d\tau = \int (\hat{A}\phi)^* \psi \, d\tau$$

In vectors, $\Rightarrow \int \phi^\dagger \hat{A} \psi \, d\tau = \int (\hat{A}\phi)^\dagger \psi \, d\tau$

Substitute the value of operator $\hat{A}$ & satisfy the condⁿ

$$A^\dagger = \hat{A}$$

then $\hat{A}$ is Hermition.

Anti/skew Hermitian operator :-

$$(\langle \phi | \hat{A} | \psi \rangle)^\dagger = -(\langle \psi | \hat{A} | \phi \rangle)$$

$$\boxed{A^\dagger = -\hat{A}}$$

All the observable quantity → mom, ang mom, K.E, P.E .... all are Hermitian operator.

Momentum operator → $\hat{p} = -i\hbar \frac{\partial}{\partial x}$

$$\boxed{\left(-i\hbar \frac{d}{dx}\right)^\dagger = -i\hbar \frac{d}{dx}}$$

$$(\phi, \hat{A}\psi) = \int_{-\infty}^{+\infty} \phi^* \left(-i\hbar \frac{d}{dx}\right) \psi \, dx$$

$$= \int^{+\infty} \phi^* \left(-i\hbar \frac{d\psi}{} \right) dx$$

(Eigen value of hermitian o/p → always real)

Date :

Page No. :

$$= -i\hbar \int_{-\infty}^{+\infty} \underset{\mathrm{I}}{\phi^{+}} \underset{\mathrm{II}}{\left(\frac{d\psi}{dx}\right)} dx$$

$$= -i\hbar \underset{\overset{\|}{0}}{\left[\phi^{+}\psi\right]_{-\infty}^{+\infty}} + i\hbar \int_{-\infty}^{+\infty} \left(\frac{d\phi^{+}}{dx}\right)\psi dx$$

(at $r \to \infty$, $\psi \to 0$)

$$= i\hbar \int_{-\infty}^{+\infty} \left(\frac{d\phi^{+}}{dx}\right) \psi \, dx$$

$$= \int \left(-i\hbar \frac{d\phi}{dx}\right)^{+} \psi \, dx$$

$$= (\hat{A}\phi, \psi)$$

So Mom. operator is hermitian

Similarly, $\left(\frac{d}{dx}\right)^{+} = -\frac{d}{dx}$

Projection Operator :- An operator $\hat{?}$ is said to be projection operator if it is Hermitian and Idempotent.

$\boxed{\hat{A}^{+} = \hat{A}}$ $\boxed{\hat{A}^{2} = A}$

For projection o/p, eigen values can be calculated,

$\hat{A}|\psi\rangle = \lambda|\psi\rangle$ $\lambda \to$ Eigen value

& $A^{2}|\psi\rangle = \lambda^{2}|\psi\rangle$

$\Rightarrow A|\psi\rangle = \lambda^{2}|\psi\rangle$

$\lambda|\psi\rangle = \lambda^{2}|\psi\rangle$

$(\lambda^{2} - \lambda)|\psi\rangle = 0$

$|\psi\rangle \neq 0$

So $\lambda^{2} - \lambda = 0 \Rightarrow \lambda(\lambda - 1) = 0$

$\Rightarrow \boxed{\lambda = 0 \text{ or } 1}$

eigen value of projection o/p are 0 or 1.

(1) The product of 2 commutator projection operators is also a projection o/p.

Consider 2 projection o/p $\hat{P}_1$ & $\hat{P}_2$

then $\hat{P}_1^+ = \hat{P}_1$ $\qquad$ $\hat{P}_2^+ = \hat{P}_2$

$\hat{P}_1^2 = \hat{P}_1$ $\qquad$ $\hat{P}_2^2 = \hat{P}_2$

Product $= (\hat{P}_1\hat{P}_2)^+ = \hat{P}_1\hat{P}_2$

$(\hat{P}_1\hat{P}_2)^2 = \hat{P}_1\hat{P}_2$

dagger is calculated in reverse order,

$(\hat{P}_1\hat{P}_2)^+ = \hat{P}_2^+\hat{P}_1^+ = \hat{P}_2\hat{P}_1$

$\neq \hat{P}_1\hat{P}_2$ Not always

($P_2P_1 = P_1P_2$ when they commute only)

Now apply the cond^n of product of commutating.

$(\hat{P}_1\hat{P}_2)^+ = \hat{P}_2^+\hat{P}_1^+ = \hat{P}_2\hat{P}_1 = \hat{P}_1\hat{P}_2$

$(\hat{P}_1\hat{P}_2)^2 = \hat{P}_1\hat{P}_2\hat{P}_1\hat{P}_2$

$= \hat{P}_1\hat{P}_2\hat{P}_2\hat{P}_1$ (if commute)

$= \hat{P}_1\hat{P}_2$

(2) The sum of 2 projection o/p is a projection o/p if they anti-commute to each other or orthogonal to each other

$(\hat{P}_1+\hat{P}_2)^+ = P_2^+ + P_1^+ = P_1 + P_2$

$(\hat{P}_1+\hat{P}_2)^2 = (P_1+P_2)(P_1+P_2)$

$= P_1^2 + P_2^2 + (P_1P_2 + P_2P_1)$

$= \hat{P}_1^2 + \hat{P}_2^2 +$ $\qquad$ it must be 0.

$(\hat{P}_1+\hat{P}_2)^2 = \hat{P}_1^2+\hat{P}_2^2$

Date :
Page No. :

Prob :- Check weather the operator is projector or not
$|\psi\rangle\langle\psi|$

$(|\psi\rangle\langle\psi|)^{\dagger} = |\psi\rangle\langle\psi|$ satisfied

Reverse → ~~bra~~ vector so → its Hermitian o/p → ~~bra~~ ket vector

$$(|\psi\rangle\langle\psi|)^2 = (|\psi\rangle\langle\psi|)(|\psi\rangle\langle\psi|)$$
$$= |\psi\rangle \langle\psi|\psi\rangle \langle\psi|$$
$$= \underbrace{\langle\psi|\psi\rangle}_{\text{must be } 1} |\psi\rangle\langle\psi|$$

must be 1 to satisfy the Idempotent cond^n
(Normalised to unity)

If $\langle\psi|\psi\rangle$ is Normalised to unity then $|\psi\rangle\langle\psi|$ is projection operator.

Inverse of an operator :-
OR
Non-singular operator :-

Matrix for which inverse exist → called Non-singular matrix. $A^{-1} = \frac{\text{adj} A}{|A|}$

Illy, Operator $\hat{A}$ for which inverse exist called the Non-singular operator.

$$\hat{A}|\psi\rangle = |v\rangle$$
$$\hat{B}|v\rangle = |u\rangle$$

In this case $\hat{B}$ is the inverse of $\hat{A}$ & $\hat{A}$ is called Non-singular o/p.

Singular operator :-

for any non-zero state vector $|u\rangle \neq 0$

Eigen value of projection o/p are 0 or 1.

(1) The product of 2 commutator projection operators is also a projection o/p.

Consider 2 projection o/p $\hat{P}_1$ & $\hat{P}_2$.

then $\hat{P}_1^{+} = \hat{P}_1$ $\quad \hat{P}_2^{+} = \hat{P}_2$

$\hat{P}_1^{2} = \hat{P}_1$ $\quad \hat{P}_2^{2} = \hat{P}_2$

Product $= (\hat{P}_1\hat{P}_2)^{+} = \hat{P}_1\hat{P}_2$

$(\hat{P}_1\hat{P}_2)^2 = \hat{P}_1\hat{P}_2$

deger is calculated in reverse order,

$(\hat{P}_1\hat{P}_2)^{+} = \hat{P}_2^{+}\hat{P}_1^{+} = \hat{P}_2\hat{P}_1$

$\neq \hat{P}_1\hat{P}_2$ Not always

($P_2P_1 = P_1P_2$ when they commute only)

Now apply the cond^n of product of commutating.

$(\hat{P}_1\hat{P}_2)^{+} = \hat{P}_2^{+}\hat{P}_1^{+} = \hat{P}_2\hat{P}_1 = \hat{P}_1\hat{P}_2$

$(\hat{P}_1\hat{P}_2)^2 = \hat{P}_1\hat{P}_2\hat{P}_1\hat{P}_2$

$= \hat{P}_1\hat{P}_1\hat{P}_2\hat{P}_1$ (if commute)

$= \hat{P}_1\hat{P}_2$

(2) The sum of 2 projection o/p is a projection o/p if they anti-comute to each other or orthogonal to each other.

$(\hat{P}_1 + \hat{P}_2)^{+} = P_2^{+} + P_1^{+} = P_1 + P_2$

$(\hat{P}_1 + \hat{P}_2)^2 = (P_1 + P_2)(P_1 + P_2)$

$= P_1^2 + P_2^2 + \underbrace{(P_1P_2 + P_2P_1)}_{\text{it must be } 0}$

$= \hat{P}_1^2 + \hat{P}_2^2 +$

$(\hat{P}_1 + \hat{P}_2)^2 = \hat{P}_1^2 + \hat{P}_2^2$

Date :
Page No. :

Prob :- Check weather the operator is projector or n
$|\psi\rangle\langle\psi|$

$$(|\psi\rangle\langle\psi|)^\dagger = |\psi\rangle\langle\psi|$$ satisfied

Reverse → bra vector so → it's Hermitian o/p → ket vector

$$(|\psi\rangle\langle\psi|)^2 = (|\psi\rangle\langle\psi|)(|\psi\rangle\langle\psi|)$$
$$= |\psi\rangle \langle\psi|\psi\rangle \langle\psi|$$
$$= \underbrace{\langle\psi|\psi\rangle}_{=} |\psi\rangle\langle\psi|$$

must be 1 to satisfy the Idempotent cond^n
(Normalised to unity)

If $\langle\psi|\psi\rangle$ is Normalised to unity then $|\psi\rangle\langle\psi|$ is projection operator.

Inverse of an operator :-
OR
Non-singular operator :-

Matrix for which inverse exist → called Non-singular matrix. $A^{-1} = \frac{\text{adj}\,A}{|A|}$
||ly, Operator $\hat{A}$ for which inverse exist called the Non-singular operator.

$$\hat{A}|\psi\rangle = |v\rangle$$
$$\hat{B}|v\rangle = |u\rangle$$

In this case $\hat{B}$ is the inverse of $\hat{A}$ & $\hat{A}$ is called Non-singular o/p.

Singular operator :-
for any non-zero state vector $u\rangle \neq 0$

$\hat{A}|\psi\rangle = 0$ state anhilate

There is no such op that $\hat{B}|0\rangle = |\psi\rangle$
So inverse of A i.e. A' not exist & $\hat{A}$ is called Singular o/p.

Inverse Exist → Non-singular o/p
Invers not exist → Singular o/p

Unitary Operator :- An operator is said to be unitary if

$$\boxed{\hat{A}^{+}\hat{A} = I}$$

or

$$\boxed{\hat{A}\hat{A}^{+} = I}$$

We know $\hat{A}A^{-1} = I$

~~So for unitary o/p~~ $\boxed{\hat{A}^{+} = A^{-1}}$

Parity operator :-
Parity means space reflection about the origin
Symmetry → space reflection in the pair

An operator is said to be parity o/p if

$$P\psi(x) = \psi(-x)$$

$$P\psi(r) = \psi(-r)$$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
$-r = -x\hat{i} - y\hat{j} - z\hat{k}$

for Cartisian ⇒ $i = -i$, $j = -j$, $k = -k$
Polar ⇒ $r \to r$, $\theta \to \pi - \theta$, $\phi \to \pi + \phi$
(radius vector cant be -ve)
cylindrical ⇒
$s \to s$, $\phi \to \pi + \phi$, $z \to -z$

Date :
Page No. :

$$\hat{P}\,\psi(x) = \psi(-x)$$

If $\psi(x)$ is eigen func^n of parity o/p,

$$\hat{P}\,\psi(x) = \lambda\,\psi(x)$$

$$P^2\,\psi(x) = \lambda^2\,\psi(x)$$

$$P(\psi(x)) = \lambda^2\,\psi(x)$$

$$\psi(x) = \lambda^2\,\psi(x)$$

$$(\lambda^2 - 1)\,\psi(x) = 0$$

$\psi(x) \neq 0$ so $\lambda^2 - 1 = 0$

$$\boxed{\lambda = \pm 1}$$

eigen value of parity o/p $\Rightarrow$ +1 & −1

for $\lambda = +1$ $\quad P\,\psi(x) = +\psi(x)$

$\Rightarrow \boxed{\psi(-x) = +\psi(x)} \Rightarrow$ Even Parity

for $\lambda = -1$ $\quad P\,\psi(x) = -\psi(x)$

$\boxed{\psi(-x) = -\psi(x)} \Rightarrow$ Odd Parity

Properties :-

(1) Eigen value of parity o/p is +1 & −1.

(2) The eigen func^n of parity operator has definate parity.
P = Even or P = odd

(3) Parity o/p is Involumatry

$$\boxed{P^2 = I}$$

(4) Parity operator is hermitian.

(5) Parity o/p commutes with hamiltonian if V(x) pot^n is symmetric.

$$[H, P] = 0$$

$$\hat{P}\,H\,\psi(x) = \hat{P}\left\{\left[\frac{p^2}{2m} + V(x)\right]\psi(x)\right\} \longrightarrow$$

$$p = -i\hbar\frac{d}{dx} \Rightarrow p^2 = \hbar^2\frac{d}{dx^2}$$

$$\hat{P}(p) = +p$$

$$\left\{\hat{P}\,x = -x \qquad \hat{P}\,dx = -dx \Rightarrow \frac{d^2}{\,} = +\frac{d^2}{\,}\right\}$$

Date :-
Page No. :

$$\Rightarrow \left\{ \frac{p^2}{2m} + V(-x) \right\} \psi(-x)$$

$\hat{P} H \psi(x) = H \psi(-x)$

$\hat{P} H \psi(x) = H \hat{P} \psi(x)$

$(PH - HP)\, \psi(x) = 0$

$[P, H] = 0 \qquad \psi(x) \neq 0$

$\left\{ \text{for symmetric pot} \quad H = \frac{p^2}{2m} + V(-x) \right.$

Physical meaning of 2 operator commute,

$[A\,B] = AB - BA \Rightarrow [A\,B] = 0$

→ If 2 operator commute they can be measure simultanously with arbitrary accuracy.

→ & (P, H) They can have simultaneous eigen fun^n, or complete set of eigen fun^n.

→ E. fun^n of P have definate parity, therefore

" " H " "

In case of symmetric pot^n, the eigen fun^n of Hamiltonian or energy eigen fun^n have definate parity (either even or odd)

{ fun^n can't be of type e^{-x} → nither even nor odd }

Problem :- Consider the states $|\psi\rangle = 9i|\phi_1\rangle + 2|\phi_2\rangle$

& $|\chi\rangle = \frac{-i}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{2}}|\phi_2\rangle$

where $|\phi_1\rangle$ & $|\phi_2\rangle$ are orthonormal basis vectors.

(i) Calculate $|\chi\rangle\langle\chi|$ & $|\chi\rangle\langle\psi|$

(ii) $|\psi\rangle\langle\psi|$ & $|\chi\rangle\langle\chi|$

(iii) $Tr(|\psi\rangle\langle\psi|)$ & $Tr(|\chi\rangle\langle\chi|)$

(iv) Are $|\psi\rangle\langle\psi|$ & $|\chi\rangle\langle\chi|$ operators projector or not?

$|\psi\rangle = 9i|\phi_1\rangle + 2|\phi_2\rangle$

$\Rightarrow \langle\psi| = 9i \ldots$

Date :

Page No. :

$$|x\rangle = \frac{-i}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{2}}|\phi_2\rangle$$

$$\Rightarrow \langle x| = \frac{i}{\sqrt{2}}\langle\phi_1| + \frac{1}{\sqrt{2}}\langle\phi_2|$$

(i) $$|\psi\rangle\langle x| = \left(9i|\phi_1\rangle + 2|\phi_2\rangle\right)\left(\frac{i}{\sqrt{2}}\langle\phi_1| + \frac{1}{\sqrt{2}}\langle\phi_2|\right)$$

$$= \frac{-9}{\sqrt{2}}|\phi_1\rangle\langle\phi_1| + \frac{9i}{\sqrt{2}}|\phi_1\rangle\langle\phi_2| + \frac{2i}{\sqrt{2}}|\phi_2\rangle$$
$$+ \frac{2}{\sqrt{2}}|\phi_2\rangle\langle\phi_2|$$

$$= \frac{-9}{\sqrt{2}}(1) + 0 + 0 + \frac{2}{\sqrt{2}}(1)$$
$$= -\frac{9}{\sqrt{2}} + \frac{2}{\sqrt{2}} = -\frac{7}{\sqrt{2}}$$

$|\phi_1\rangle\langle\phi_1| \Rightarrow$ operator $\neq 1$
Not Scalar fun^n

$$|x\rangle\langle\psi| = \left(|\psi\rangle\langle x|\right)^\dagger$$

$$|x\rangle\langle\psi| = \frac{-9}{\sqrt{2}}|\phi_1\rangle\langle\phi_1| - \frac{9i}{\sqrt{2}}|\phi_2\rangle\langle\phi_1| - \frac{2i}{\sqrt{2}}|\phi_1\rangle\langle\phi_2|$$
$$+ \frac{2}{\sqrt{2}}|\phi_2\rangle\langle\phi_2|$$

(ii) $$|\psi\rangle\langle\psi| = \left(9i|\phi_1\rangle + 2|\phi_2\rangle\right)\left(-9i\langle\phi_1| + 2\langle\phi_2|\right)$$

$$= +81|\phi_1\rangle\langle\phi_1| + 18i|\phi_1\rangle\langle\phi_2| - 18i|\phi_2\rangle\langle\phi_1|$$
$$+ 4|\phi_2\rangle\langle\phi_2|$$

$$|x\rangle\langle x| = \left(|\psi\rangle\langle\psi|\right)^\dagger$$

$$= \frac{1}{2}|\phi_1\rangle\langle\phi_1| \mp \frac{i}{2}|\phi_1\rangle\langle\phi_2| + \frac{i}{\sqrt{2}}|\phi_2\rangle\langle\phi_1| + \frac{1}{2}|\phi_2$$

(iii) Here 2 basis vectors are there so take matrix of order 2×2.

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$= 81(1) - 0 + 0 + 4(1) = 85$

$Tr \langle x|x\rangle = \frac{1}{2}\langle\phi_1|\phi_2\rangle - \frac{i}{\sqrt{2}}\langle\phi_2|\phi_1\rangle + \frac{i}{\sqrt{2}}\langle\phi_1|\phi_2\rangle + \frac{1}{2}\langle\phi_2|\phi_2\rangle$

$= \frac{1}{2} + \frac{1}{2} = 1$

Date:

Page No.:

$A_{11} = \langle\phi_1|\hat{A}|\phi_1\rangle$

$A_{12} = \langle\phi_1|\hat{A}|\phi_2\rangle$

$A_{21} = \langle\phi_2|\hat{A}|\phi_1\rangle$

$A_{22} = \langle\phi_2|\hat{A}|\phi_2\rangle$

Trace is cyclic i.e. $Tr(AB) = Tr(BA)$

$Tr(|\psi\rangle\langle\psi|) = Tr(\langle\psi|\psi\rangle) = 81 + 4$

$= 85$

$Tr(|x\rangle\langle x|) = Tr(\langle x|x\rangle) = 1$

Directly

if $|\psi\rangle$ is normalised then

$|\psi\rangle = 9i|\phi_1\rangle + 2|\phi_2\rangle$

$= |9i|^2 + |2|^2$

$= 9i(-i) + 4$

$= 81(-i^2) + 4$

$= 81 + 4 = 85$

$Tr(\langle\psi|\psi\rangle) = 85$

& $|x\rangle = \frac{-i}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{2}}|\phi_2\rangle$

$Tr(\langle x|x\rangle) = \frac{i(-i)}{2} + \frac{1}{2}$

$= \frac{1}{2} + \frac{1}{2} = 1$

(iv) $|\psi\rangle\langle\psi|$ & $|x\rangle\langle x|$

If w.fuc^n is normalised to unity then it'll be projector o/p.

$(|\psi\rangle\langle\psi|)^\dagger = (|\psi\rangle\langle\psi|)$

$(|\psi\rangle\langle\psi|)^2 = |\psi\rangle\langle\psi|$

$|\psi\rangle\langle\psi|$ is not normalised to unity $\quad \langle\psi|\psi\rangle \neq 1$

$|x\rangle\langle x|$ is ~~not~~ " " " $\quad \langle x|x\rangle = 1$

So $|\psi\rangle\langle\psi| \rightarrow$ Not Projection o/p.

$|x\rangle\langle x| \rightarrow$ Projection o/p.

Prob :- Check weather this operator $\frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$ is projector or not.

$$\left(\frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}\right)^\dagger = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} \quad (\langle\psi|\psi\rangle \rightarrow no)$$

$$\left(\frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}\right)^2 = \frac{(|\psi\rangle\langle\psi|)(|\psi\rangle\langle\psi|)}{\langle\psi|\psi\rangle\langle\psi|\psi\rangle} = \frac{|\psi\rangle\cancel{\langle\psi|\psi\rangle}\langle\psi|}{\cancel{\langle\psi|\psi\rangle}\langle\psi|\psi\rangle}$$

$$= \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$$

This is P.O. either ψ is normalised ...

Date :
Page No. :

Que N = b^+b, where the operator $\hat{b}$ & $\hat{b}^+$ satisfie relation $\hat{b}\hat{b}^+ + \hat{b}^+\hat{b} = 1$ and $\hat{b}^2 = 0$ & $(\hat{b}^+)^2 =$
N is also a operator. The eigen values of N are
(a) +ve & −ve integers
(b) ~~All +ve~~ integers
(c) ~~±1 & 0 only~~
(d) 0 & 1 only

$$N = b^+b \Rightarrow N\psi = b^+b\,\psi$$

$$\hat{N}\psi(x) = \lambda\psi(x)$$

$$b^+b|\psi\rangle = \lambda|\psi\rangle$$

Again, $(b^+b)^2|\psi\rangle = \lambda^2|\psi\rangle$

$$b^+b\,b^+b|\psi\rangle = \lambda^2|\psi\rangle$$

$$\Rightarrow b^+(1-b^+b)\,b|\psi\rangle = \lambda^2|\psi\rangle$$

$$\Rightarrow (b^+b - b^+b^+bb)|\psi\rangle = \lambda^2|\psi\rangle$$

$$\Rightarrow (b^+b - b^{+2}b^2)|\psi\rangle = \lambda^2|\psi\rangle$$

$$\Rightarrow b^+b|\psi\rangle = \lambda^2|\psi\rangle$$

$$\Rightarrow \lambda|\psi\rangle = \lambda^2|\psi\rangle$$

$$\Rightarrow (\lambda-\lambda^2)|\psi\rangle = 0$$

$$\Rightarrow \lambda-\lambda^2 = 0$$

$$\Rightarrow \lambda(1-\lambda) = 0$$

$$\boxed{\lambda = 0, 1}$$ Ans

Que: A particle in the infinite potⁿ well

$$V(x) = \begin{cases} 0, & 0<x<a \\ \infty, & \text{otherwise} \end{cases}$$

has the initial wave fucⁿ, $\psi(x,0) = A\sin^3\left(\frac{\pi x}{a}\right)$,
~~$0 \le x \le a$~~

(i) ~~Determine the value of A so that $\psi(x)$ is normalis~~

Date:
Page No.:

(ii) Calculate the expectation value of x as a fun^n of time t.

$\langle x \rangle = ?$

$$\psi(x,0) = A \sin^3\left(\frac{\pi x}{a}\right)$$

We have, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\Rightarrow \sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$$

$$\psi(x,0) = \frac{A}{4}\left[3\sin\left(\frac{\pi x}{a}\right) - \sin\left(\frac{3\pi x}{a}\right)\right]$$

$$= \sqrt{\frac{a}{2}}\sqrt{\frac{2}{a}}\,\frac{A}{4}\left[3\sin\left(\frac{\pi x}{a}\right) - \sin\left(\frac{3\pi x}{a}\right)\right]$$

$$= \frac{A}{4}\sqrt{\frac{a}{2}}\left[3\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right) - \sqrt{\frac{2}{a}}\sin\left(\frac{3\pi x}{a}\right)\right]$$

$$\psi(x,0) = \frac{A}{4}\sqrt{\frac{a}{2}}\left[3\phi_1 - \phi_3\right]$$

$$|\psi(x,0)\rangle = \frac{A}{4}\sqrt{\frac{a}{2}}\left[3|\phi_1\rangle - |\phi_3\rangle\right]$$

Normalisation, $\left|\frac{3}{4}A\sqrt{\frac{a}{2}}\right|^2 + \left|\frac{A}{4}\sqrt{\frac{a}{2}}\right|^2 = 1$

$$\frac{9}{16}A^2\frac{a}{2} + \frac{A^2}{16}\frac{a}{2} = 1$$

$$\frac{A^2}{16}\frac{a}{2}(9+1) = 1 \Rightarrow \frac{A^2}{16}\frac{a}{\cancel{2}}(\cancel{10}^{5}) = 1$$

$$A^2 = \frac{16}{5a}$$

$$A = \frac{4}{\sqrt{5a}}$$

So $\psi(x,0) = \frac{4}{\sqrt{5a}}\sin^3\left(\frac{\pi x}{a}\right)$

$$\Rightarrow |\psi(x,0)\rangle = \frac{\cancel{A}\,4}{\sqrt{5a}}\sqrt{\frac{a}{2}}\left[3|\phi_1\rangle - |\phi_3\rangle\right]$$

$$= \frac{1}{\sqrt{10}}\left[3|\phi_1\rangle - |\phi_3\rangle\right]$$

$$|\psi(x,t)\rangle = \frac{3}{\sqrt{10}}e^{-\frac{iE_1 t}{\hbar}}|\phi_1\rangle - \frac{1}{\sqrt{10}}e^{-\frac{iE_3 t}{\hbar}}|\phi_3\rangle$$

Date :
Page No. :

$\langle x \rangle = \langle \psi(t) | \hat{x} | \psi(t) \rangle$

Que: A particle is in the normalised state $|\psi\rangle$ which is in the superposition of energy eigen states $|E_0 = 10\,eV\rangle$ & $|E_1 = 30\,eV\rangle$. The average value of the energy of the particle in the state $|\psi\rangle$ i.e. $\langle \psi | H | \psi \rangle = 20\,eV = \langle E \rangle$.

The state is given by $|\psi\rangle =$

(a) $\frac{1}{2} |E_0 = 10\,eV\rangle + \frac{\sqrt{3}}{4} |E_1 = 30\,eV\rangle$

(b) $\frac{1}{\sqrt{3}} |E_0 = 10\,eV\rangle + \sqrt{\frac{2}{3}} |E_1 = 30\,eV\rangle$

(c) $\frac{1}{2} |E_0 = 10\,eV\rangle - \frac{\sqrt{3}}{4} |E_1 = 30\,eV\rangle$

(d) $\frac{1}{\sqrt{3}} |E_0 = 10\,eV\rangle - \sqrt{\frac{2}{3}} |E_1 = 30\,eV\rangle$

(e) $\frac{1}{\sqrt{2}} |E_0 = 10\,eV\rangle - \frac{1}{\sqrt{2}} |E_1 = 30\,eV\rangle$

Date :

Page No. :

(f) $\frac{1}{\sqrt{2}} |E_0 = 10 eV\rangle + \frac{1}{\sqrt{2}} |E_1 = 30 eV\rangle$

$|\psi\rangle$ is superposition of $|\phi_0\rangle$ & $|\phi_1\rangle$ $\left\{ \begin{array}{l} |\phi_0\rangle = |E_0 = 10 eV\rangle \\ |\phi_1\rangle = |E_1 = 30 eV\rangle \end{array} \right.$

$|\psi\rangle = C_0 |\phi_0\rangle + C_1 |\phi_1\rangle$

$|C_0|^2 + |C_1|^2 = 1$ —— (1)

$20 = \langle\psi|H|\psi\rangle \Rightarrow |C_0|^2 \times 10 \text{ eV} + |C_1|^2 \times 30 \text{ eV} = 2$

$\Rightarrow |C_0|^2 + 3|C_1|^2 = 2$ —— (2)

from (1) & (2) $\Rightarrow$ $\cancel{|C_0|^2} + |C_1|^2 = 1$

$\cancel{|C_0|^2} + 3|C_1|^2 = 2$

$-2|C_1|^2 = -1 \Rightarrow |C_1|^2 = \frac{1}{2}$

$|C_0|^2 = 1 - |C_1|^2 = 1 - \frac{1}{2} = \frac{1}{2}$

i.e. $|C_1| = |C_2| = \frac{1}{\sqrt{2}}$

Also by Checking (a) $\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2 = \frac{1}{4} + \frac{3}{16} \neq 1$

(e) $\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-i}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$

(f) $\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$

(e) $\left|\frac{1}{\sqrt{2}}\right|^2 \times 10 + \left|\frac{-i}{\sqrt{2}}\right|^2 \times 30$

$\frac{1}{2} \times 10 + \frac{1}{2} \times 30 = 5 + 15 = 20$

(f) $\left|\frac{1}{\sqrt{2}}\right|^2 \times 10 + \left|\frac{1}{\sqrt{2}}\right|^2 \times 30$

$\frac{1}{2} \times 10 + \frac{1}{2} \times 30 = 5 + 15 = 20$

Date :
Page No. :

Que :- Consider a system whose initial state $|\psi(0)\rangle$ & hamiltonian H are given by

$$|\psi(0)\rangle = \frac{1}{5}\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \qquad H = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{pmatrix}$$

(i) If a measurment of energy is carried out, what values will be obtain & with what probabilities.
(ii) Find the state of the system at any later time
(iii) Find the expectation value of the energy at t = 0 & at any later time t.

The eigen value of this hamiltonian are the possible value of energy.

$$H = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{bmatrix}$$

$$|H - \lambda I| = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & -\lambda & 5 \\ 0 & 5 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)[(-\lambda)(-\lambda) - 25] = 0$$

$$\Rightarrow (3-\lambda)(\lambda^2 - 25) = 0$$

$$\Rightarrow \lambda = 5, -5, 3$$

Eigen values, $\lambda = 3, 5, -5$

$E_1 = 3$, $E_2 = 5$, $E_3 = -5$

for $\lambda = 3$,

$$\begin{bmatrix} 3-3 & 0 & 0 \\ 0 & -3 & 5 \\ 0 & 5 & -3 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 5 \\ 0 & 5 & -3 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3|\phi_2\rangle + 5|\phi_3\rangle = 0$$

$$5|\phi_2\rangle - 3|\phi_3\rangle = 0$$

Date :
Page No. :

for, $E_1 = 3$, $H|\phi_1\rangle = E_1|\phi_1\rangle$ $\quad |\phi_1\rangle = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 3\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$3a + 0b + 0c = 3a$

$0 + 0 + 5c = 3b \Rightarrow c = \frac{3b}{5}$

$0 + 5b + 0 = 3c$

$5b = 3c \Rightarrow 5b = 3 \cdot \frac{3b}{5}$

$25b = 9b \Rightarrow 16b = 0 \Rightarrow b = 0$

Also $c = 0$

$|\phi_1\rangle = A\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$ (it may be $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} -i \\ 0 \\ 0 \end{pmatrix}$ or any .... multi. by a no. normalised to unity

$\langle\phi_1| = A^*[a^* \; 0 \; 0]$

$\langle\phi_1|\phi_1\rangle = 1$

$\Rightarrow |A|^2 |a|^2 = 1 \Rightarrow |A|^2 = \frac{1}{|a|^2}$

$\Rightarrow A = \frac{1}{a}$ so $|\phi_1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

IIly

for $E_2 = 5$, $H|\phi_2\rangle = E_2|\phi_2\rangle$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 5\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$3a = 5a$

$5c = 5b \Rightarrow \frac{c}{b} = 1 \Rightarrow c = b$

$5b = 5c$

$\Rightarrow 5c = 5c \Rightarrow c = 1$

so $b = c = 1$

$|\phi_2\rangle = \frac{1}{}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Date :
Page No. :

$E_3 = -5$, $\quad |\phi_3\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ +1 \end{bmatrix}$

$$|\psi(0)\rangle = C_1|\phi_1\rangle + C_2|\phi_2\rangle + C_3|\phi_3\rangle$$

$$\frac{1}{5}\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = C_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{C_2}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \frac{C_3}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ +1 \end{bmatrix}$$

$$C_1 = \frac{3}{5}$$

$$\begin{cases} \frac{C_2}{\sqrt{2}} - \frac{C_3}{\sqrt{2}} = 0 \\ \frac{C_2}{\sqrt{2}} + \frac{C_3}{\sqrt{2}} = \frac{4}{5} \end{cases}$$

$$2\frac{C_2}{\sqrt{2}} = \frac{4}{5} \Rightarrow C_2 = \frac{2\sqrt{2}}{5}$$

$$\& \quad C_3 = \frac{2\sqrt{2}}{5}$$

So $|\psi(0)\rangle = \frac{3}{5}|\phi_1\rangle + \frac{2\sqrt{2}}{5}|\phi_2\rangle + \frac{2\sqrt{2}}{5}|\phi_3\rangle$

Prob. of state $|\phi_1\rangle = \left|\frac{3}{5}\right|^2 \Rightarrow P_1 = \frac{9}{25}$

$|\phi_2\rangle = \left|\frac{2\sqrt{2}}{5}\right|^2 \Rightarrow P_2 = \frac{8}{25}$

$|\phi_3\rangle = \left|\frac{2\sqrt{2}}{5}\right|^2 \Rightarrow P_3 = \frac{8}{25}$

(iii), $\langle E\rangle = ?$

$$\langle E(0)\rangle = \langle\psi(0)|H|\psi(0)\rangle$$

$$= \frac{1}{5}\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{bmatrix}\frac{1}{5}\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

$$= \frac{1}{25}$$

$$\langle E(0)\rangle = \frac{27}{25} = \langle E(t)\rangle$$

Date :

Page No. :

$$|\psi(t)\rangle = \frac{3}{5} e^{-\frac{i3t}{\hbar}} |\phi_1\rangle + \frac{2\sqrt{2}}{5} e^{-\frac{i5t}{\hbar}} |\phi_2\rangle + \frac{2\sqrt{2}}{5} e^{\frac{i5t}{\hbar}} |\phi$$

Non - Degenerate Case :-

Let $|\phi_1\rangle$ be the eigenfuc^n of $\hat{A}$ with eigen value a.

If $\hat{A}$ & $\hat{B}$ commute then

$$[\hat{A}\ \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = 0$$

Eigen value eq^n, $\hat{A}|\phi_1\rangle = a|\phi_1\rangle$

and $\hat{A}\hat{B}|\phi_1\rangle = \hat{B}\hat{A}|\phi_1\rangle$ ($\hat{A}$ & $\hat{B}$ commute

$$= \hat{B}\, a|\phi_1\rangle$$

$$\hat{A}(\hat{B}|\phi_1\rangle) = a(\hat{B}|\phi_1\rangle)$$

According this eq^n, $\hat{B}|\phi_1\rangle$ is an eigen state of operator $\hat{A}$ but this will linearly depand on state $|\phi_1\rangle$ bcoz for Non-degenerate case, there will be only 1 state possib If other state possible then it may depand on $|\phi_1\rangle$.

i.e. $\boxed{\hat{B}|\phi_1\rangle = b|\phi_1\rangle}$

If 2 operator commute, they have simultaneous eigen fuc^n. i.e. eigen value is different but eigen fuc^n is same

Degenerate Case :- for 2 fold degenerate case, 2 states $|\phi_1\rangle$ & $|\phi_2\rangle$ be the eigen fuc^n of $\hat{A}$ with eigen value a.

$$\hat{A}|\phi_1\rangle = a|\phi_1\rangle$$
$$\hat{A}|\phi_2\rangle = a|\phi_2\rangle$$

Any linear combination of these 2 states $|\phi_1\rangle$ & $|\phi_2\rangle$ will have same eigen value. $c_1|\phi_1\rangle + c_2|\phi_2\rangle$

Le- an another o/p $\hat{B}$,

$$\hat{A}(\hat{B}|\phi_1\rangle) = \hat{B}\hat{A}|\phi_1\rangle$$
$$= a(\hat{B}|\phi_1\rangle)$$

$\hat{B}|\phi_1\rangle$ be the eigen fuc^n of $\hat{A}$ with eigen value of a;

$$\boxed{|\phi_2\rangle = \hat{B}|\phi_1\rangle \neq b|\phi_1\rangle}$$

eigen fuc^n of A $\neq$ eigen fuc^n of B.

$$\left\{ \begin{aligned} \hat{A}(c_1|\phi_1\rangle + c_2|\phi_2\rangle) &= c_1 \hat{A}|\phi_1\rangle + c_2 \hat{A}|\phi_2\rangle \\ &= c_1 a|\phi_1\rangle + c_2 a|\phi_2\rangle \\ &= a(c_1|\phi_1\rangle + c_2|\phi_2\rangle) \end{aligned} \right\}$$

e.g. 2 different eigen states $A\sin kx$, $B\cos kx$ gives same energy eigen value $\frac{\hbar^2 k^2}{2m}$.

for m-fold degenerate, Out of ∞ no. of eigen states, at least m eigen fuc^n exist for m eigen states.

$\Rightarrow$ If 2 Operator $\hat{A}$ and $\hat{B}$ commute each other then each eigen fuc^n of operator $\hat{A}$ will always be the eigen fuc^n of operator $\hat{B}$ with non-degenerate eigen value & vice-versa.

$\Rightarrow$ If 2 operator commute then each eigen fuc^n of operator $\hat{A}$ with degenerate eigen value in generalo no: be the eigen fuc^n of operator $\hat{B}$.

But we can construct the eigen fuc^n of operator $\hat{B}$ with the help of eigen fuc^n of operator $\hat{A}$ & vice-versa.

If more than 2 o/p ($\hat{A}$, $\hat{B}$, $\hat{C}$ ----) exist then they have same eigen fuc^n.

Properties of Commutator :-

(1) Antisymmetry :-

$$[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$$

(2) Linearity :-

$$[\hat{A}, \hat{B} + \hat{C} + \hat{D} + \cdots] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}] + [\hat{A}, \hat{D}] + \cdots$$

(3) Hermitian conjugate of Commutator :-

$$[\hat{A}\ \hat{B}]^{+} = [\hat{B}^{+}, \hat{A}^{+}]$$

Imp (4) Distributive Property :-

$$[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A},\hat{C}] + [\hat{A},\hat{B}]\hat{C}$$

(5) Jacobi Identity :-

$$[\hat{A},[\hat{B},\hat{C}]] + [\hat{B},[\hat{C},\hat{A}]] + [\hat{C},[\hat{A},\hat{B}]] = 0$$

(6) Trace of Commutator :-

$$Tr[\hat{A},\hat{B}] = 0$$

(7) $[\hat{A}, f(\hat{A})] = 0$ any fun^n of o/p commute with o/p itself.

(8) $[f(\hat{A}), G(\hat{A})] = 0$

$[f(\hat{A}), G(\hat{B})] \neq 0$

$[f(\hat{A}), G(\hat{B})] = 0$ only if $[\hat{A},\hat{B}] = 0$

(9) $[\hat{A}, \hat{B}^{n}] = n\hat{B}^{n-1}[\hat{A},\hat{B}]$

(10) $[\hat{A}^{n}, B] = n\hat{A}^{n-1}[\hat{A},\hat{B}]$

(11) $[x_i, x_j] = 0$

$[x_i, p_j] = i\hbar\delta_{ij}$

$[p_i, p_j] = 0$

Problem :- $\hat{A}$ & $\hat{B}$ are 2 quantum mechanical operators if $[A, B]$ stands for the commutator of A & B then the value of $[[\hat{A}, \hat{B}], [\hat{B}, \hat{A}]]$ is equal to

(a) $\hat{A}\hat{B}\hat{A}\hat{B} - \hat{B}\hat{A}\hat{B}\hat{A}$

(b) $\hat{A}(\hat{A}\hat{B} - \hat{B}\hat{A}) - \hat{B}(\hat{B}\hat{A} - \hat{A}\hat{B})$

(c) $([\hat{A}\ \hat{B}])^2$

✓(d) 0

$$[[\hat{A}, \hat{B}], [\hat{B}\ \hat{A}]] = -[[A, B], [A, B]]$$
$$= 0$$

OR $[[\hat{A}, \hat{B}], [\hat{B}\ \hat{A}]] = [(AB - BA), (BA - AB)]$

$$= (AB - BA)(BA - AB) - (BA - AB)(AB - BA)$$
$$= 0$$

Que :- $[x^2, p^2]$ is

(1) $2i\hbar xp$

✓(2) $2i\hbar(xp + px)$

(3) $2i\hbar px$

(4) $2i\hbar(xp - px)$

* If $\hat{A}$ & $\hat{B}$ are two observables then the generalise expression for the uncertainity relation,

$$\boxed{(\Delta A)^2 (\Delta B)^2 \geq -\frac{1}{4}\left(\langle [\hat{A}, \hat{B}] \rangle\right)^2}$$

$\Delta A \rightarrow$ uncertainity in A

$\Delta B \rightarrow$ " " B

Equation of motion in Quantum Mechanics :-

If 2 operators commute $[\hat{A}\ \hat{B}]=0$, they are called Compitable operators.

eq^n of motion, $H|\psi\rangle = E|\psi\rangle$

$$H|\psi\rangle = i\hbar \frac{\partial|\psi\rangle}{\partial t} = i\hbar\left|\frac{\partial\psi}{\partial t}\right\rangle$$

$\left(\begin{array}{l} H^{+}=H \\ \hat{H} \text{ is hermitian o/p} \end{array}\right)$ $\langle\psi|H = -i\hbar\left\langle\frac{\partial\psi}{\partial t}\right|$

$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$ If ψ is normalised to unity then this is the expactation value of $\hat{A}$.

$$\frac{d}{dt}\langle\hat{A}\rangle = \left\langle\frac{\partial\psi}{\partial t}\right|\hat{A}|\psi\rangle + \langle\psi|\frac{\partial\hat{A}}{\partial t}|\psi\rangle + \langle\psi|\hat{A}|\frac{\partial\psi}{\partial t}\rangle$$

$$= -\frac{1}{iH}\langle\psi|\hat{H}\hat{A}|\psi\rangle + \left\langle\frac{\partial\hat{A}}{\partial t}\right\rangle + \frac{1}{i\hbar}\langle\psi|\hat{A},\hat{H}|\psi\rangle$$

$$= \left\langle\frac{\partial\hat{A}}{\partial t}\right\rangle + \frac{1}{i\hbar}\langle\psi|[\hat{A},\hat{H}]|\psi\rangle$$

$$\boxed{\frac{d}{dt}\langle\hat{A}\rangle = \left\langle\frac{\partial\hat{A}}{\partial t}\right\rangle + \frac{1}{i\hbar}\langle[\hat{A},\hat{H}]\rangle}$$

- If o/p $\hat{A}$ does not depend on time explicitely then

$$\left\langle\frac{\partial\hat{A}}{\partial t}\right\rangle = 0$$

$$\Rightarrow \boxed{\frac{d}{dt}\langle\hat{A}\rangle = \frac{1}{i\hbar}\langle[\hat{A}\ \hat{H}]\rangle}$$

- If o/p $\hat{A}$ does not depend on time explicitely & commute with hamiltonian then the corresponding dynamical variable will be constant of motion or conserved.

$$\frac{d}{dt}\langle\hat{A}\rangle = 0$$

$$\boxed{\langle\hat{A}\rangle = \text{Conserved}}$$

or $\boxed{\langle\hat{A}\rangle = \text{Constant}}$

Problems

Q.1 :- Prove that for 2 operators $\hat{A}$ & $\hat{B}$

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A}\hat{B}]]] + \cdots$$

Q.2 :- If $\hat{A}$ & $\hat{B}$ are 2 operators s.t. where

$[\hat{A}, \hat{B}] = \alpha\hat{B}$ where α is a no. then

show that $e^{\hat{A}}\hat{B}e^{-\hat{A}} = e^{\alpha}\hat{B}$

Q.3 :- If $\hat{A}$ & $\hat{B}$ are 2 operators which both commute with their commutator then prove that

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}$$

Q.1 :- Let $f(\lambda) = e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}$ $\Rightarrow f(0) = \hat{B}$

$$f'(\lambda) = e^{\lambda\hat{A}}\hat{A}\hat{B}e^{-\lambda\hat{A}} + e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}(-\hat{A})$$

$$\Rightarrow f'(0) = \hat{A}\hat{B} + \hat{B}(-\hat{A}) = \hat{A}\hat{B} - \hat{B}\hat{A} = [\hat{A},\hat{B}]$$

$$f''(\lambda) = [e^{\lambda\hat{A}}\hat{A}\hat{A}\hat{B}e^{-\lambda\hat{A}} + e^{\lambda\hat{A}}\hat{A}\hat{B}e^{-\lambda\hat{A}}(-\hat{A})]$$
$$+[e^{\lambda\hat{A}}\hat{A}\hat{B}e^{-\lambda\hat{A}}(-\hat{A}) + e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}(-\hat{A})(-\hat{A})]$$

$$\Rightarrow f''(0) = \hat{A}[\hat{A}\hat{B} - \hat{B}\hat{A}] - [\hat{A}\hat{B} - \hat{B}\hat{A}]\hat{A}$$
$$= \hat{A}[\hat{A}\hat{B}] - [\hat{A}\hat{B}]\hat{A}$$
$$= [\hat{A}, [\hat{A},\hat{B}]]$$

By Taylor series,

$$f(\lambda) = f(0) + \lambda f'(0) + \frac{\lambda^2}{2!}f''(0) + \cdots$$

$$e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}} = \hat{B} + \lambda[\hat{A}\hat{B}] + \frac{\lambda^2}{2!}[\hat{A},[\hat{A}\hat{B}]] + \cdots$$

for $\lambda = 1$

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \cdots \quad \text{Ans}$$

(1)

Q2 :- eqn (1) $\Rightarrow$

$$e^{A}\hat{B}e^{-A} = \hat{B} + \alpha\hat{B} + \frac{1}{2!}[A, \alpha B] + \frac{1}{3!}[A, [A, \alpha B]]$$

$$= \hat{B} + \alpha\hat{B} + \frac{1}{2!}\alpha[A, B] + \frac{1}{3!}[A, \alpha[A, B]]$$

$$= \hat{B} + \alpha\hat{B} + \frac{\alpha}{2!}[A, B] + \frac{\alpha}{3!}[A, [A, B]]$$

$$= \hat{B} + \alpha\hat{B} + \frac{\alpha}{2!}\alpha B + \frac{\alpha}{3!}[A, \alpha B]$$

$$= \hat{B} + \alpha\hat{B} + \frac{\alpha^2}{2!}B + \frac{\alpha^2}{3!}[A, B]$$

$$= \hat{B} + \alpha\hat{B} + \frac{\alpha^2}{2!}B + \frac{\alpha^2}{3!}(\alpha\hat{B}) = \hat{B} + \alpha\hat{B} + \frac{\alpha^2}{2!}B + \frac{\alpha^3}{3!}B$$

$$= \left(1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \cdots\right)B$$

$$\boxed{e^{A}\hat{B}e^{-A} = e^{\alpha}\hat{B}}$$ Ans

Q.3 :-

1 Dimensional Harmonic Oscillator
OR
Linear Harmonic Oscillator :-

Potⁿ for L.H.O, is $V = \frac{1}{2} K x^2$

In terms of angular freq. $V = \frac{1}{2} m\omega^2 x^2$ $\qquad \omega = \sqrt{\frac{K}{m}}$

Hamiltonian, $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$

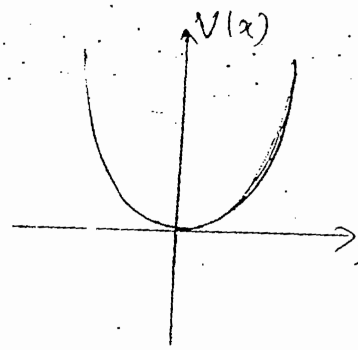
$$= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2$$

Schroodinger eqⁿ, $H\psi = E\psi$

$$\Rightarrow \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2\right]\psi(x) = E\psi$$

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left[E - \frac{1}{2}m\omega^2 x^2\right]\psi = 0$$

K is not constant, it is variable.



Hamiltonian can be written as $\hat{p}$ & $\hat{x}$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

$$\frac{\hat{H}}{\hbar\omega} = \left(\frac{1}{2}\frac{m\omega}{\hbar}\right)\hat{x}^2 + \left(\frac{1}{2m\hbar\omega}\right)\hat{p}^2$$

$$\frac{\hat{H}}{\hbar\omega} = \left(\sqrt{\frac{m\omega}{2\hbar}}\right)^2 \hat{x}^2 + \left(\frac{1}{\sqrt{2m\hbar\omega}}\right)^2 \hat{p}^2$$

← Remember

Define $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + i\frac{\hat{p}}{\sqrt{2\hbar m\omega}}$ Raising o/p

$\hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} - i\frac{\hat{p}}{\sqrt{2\hbar m\omega}}$ Lowering o/p

Raising & lowering o/p are Hermitian conjugate of each other.

$$\hat{a}^+\hat{a} = \frac{m\omega}{2\hbar}\hat{x}\hat{x} + i\frac{1}{2\hbar}\hat{x}\hat{p} - i\frac{1}{2\hbar}\hat{p}\hat{x} + \frac{1}{2\hbar m\omega}\hat{p}\hat{p}$$

$$= \frac{m\omega}{2\hbar}\hat{x}^2 + \frac{1}{2\hbar m\omega}\hat{p}^2 + \frac{i}{2\hbar}(\hat{x}\hat{p} - \hat{p}\hat{x})$$

$$= \frac{m\omega}{2\hbar}\hat{x}^2 + \frac{1}{2\hbar m\omega}\hat{p}^2 + \frac{i}{2\hbar}[\hat{x},\hat{p}]$$

$$\hat{a}^+\hat{a} = \frac{1}{\hbar\omega}\left[\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2\right] + \frac{i}{2\hbar}(+i\hbar)$$

$$= \frac{1}{\hbar\omega}\left[\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2\right] - \frac{1}{2}$$

$$= \frac{\hat{H}}{\hbar\omega} - \frac{1}{2}$$

$$\boxed{\hat{H} = \left[\hat{a}^+\hat{a} + \frac{1}{2}\right]\hbar\omega}$$

$[H, \hat{a}^+\hat{a}] \neq 0$ or $= 0$

$$\left[(\hat{a}^+\hat{a} + \tfrac{1}{2})\hbar\omega,\ \hat{a}^+\hat{a}\right]$$

$$\Rightarrow (a^+a + \tfrac{1}{2})\hbar\omega(a^+a) - (a^+a)(\hat{a}^+a + \tfrac{1}{2})\hbar\omega$$

$$\Rightarrow \hbar\omega\left[(a^+a)(a^+a) + \cancel{\tfrac{1}{2}(a^+a)} - (a^+a)(a^+a) - \cancel{(a^+a)\tfrac{1}{2}}\right]$$

$$\hbar\omega[0] = 0$$

$$\boxed{[\hat{H}, \hat{a}^+\hat{a}] = 0}$$

$$\Rightarrow [\hat{a}, \hat{a}^+] = \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = \frac{\hat{H}}{\hbar\omega} + \frac{1}{2} - \frac{\hat{H}}{\hbar\omega} + \frac{1}{2}$$

$$[\hat{a}, \hat{a}^+] = 1$$

$$\Rightarrow [\hat{a}^+, \hat{a}] = a^+a - aa^+$$

$$[\hat{a}^+, \hat{a}] = -1$$

$$\Rightarrow [\hat{a}^+\hat{a}, \hat{a}] = a^+[a, a] + [a^+, a]a =$$

$$[a^+a, a] = -a$$

$$\Rightarrow [\hat{a}^+\hat{a}, \hat{a}^+] = \hat{a}^+[\hat{a}, \hat{a}^+] + [\hat{a}^+ \hat{a}^+]\hat{a}$$

$$[\hat{a}^+\hat{a}, \hat{a}^+] = \hat{a}^+$$

We have $[\hat{H}, \hat{a}^+\hat{a}] = 0$

Let $|n\rangle$ be the eigen func^n, a^+a operator & $n \rightarrow$ eigen value

$$\hat{a}^+\hat{a}\,|n\rangle = n\,|n\rangle$$

$$\hat{H}\,|n\rangle = (\hat{a}^+\hat{a} + \tfrac{1}{2})\hbar\omega\,|n\rangle$$

$$E_n\,|n\rangle = (n + \tfrac{1}{2})\hbar\omega\,|n\rangle$$

$$\boxed{E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega} \qquad n = 0, 1, 2, 3, \ldots$$

Energy eigen value of nth state of harmonic oscillator.

for H.O., ground state energy for $n = 0$

$$\boxed{E_0 = \tfrac{1}{2}\hbar\omega}$$

In Q.M., bound state energy in ground state $\neq 0$ due to the concequence of Heisenberg Uncertainity principle.

{ In 1 Dim particle in box, $E_n = \dfrac{n^2\pi^2\hbar^2}{2ma^2}$ for, ground state energy $\neq 0$ bcoz if $E = 0$, $\Delta p = 0$ then $\Delta x = \infty$ $\quad \Delta x\,\Delta p \geq \frac{\hbar}{2}$ }

T.E. = K.E.

But for harmonic oscillator T.E. = K.E. + P.E.

here if Total energy $= 0$ then mom. $\neq 0$

if $E = 0$, then $\Delta E = 0$

$$\Delta E\,\Delta t \geq \hbar/2$$

$$\Delta t = \infty$$

If particle cross its mean position then it will come back after ∞ time.

$$H|n\rangle = E_n|n\rangle$$

$$\hat{H}\hat{a}|n\rangle$$

$$[\hat{H}, \hat{a}] = \hat{H}\hat{a} - \hat{a}\hat{H}$$

$$[(\hat{a}^+a + \tfrac{1}{2})\hbar\omega, \hat{a}] = [\hat{a}^+a, \hat{a}]\hbar\omega$$

$$= \hat{a}^+[\hat{a}\,\hat{a}]\hbar\omega + [\hat{a}^+\hat{a}]\,\hat{a}\hbar\omega$$

$$= 0 + -\hat{a}\hbar\omega = -\hat{a}\hbar\omega$$

$$\hat{H}\hat{a} - \hat{a}\hat{H} = -\hat{a}\hbar\omega$$
$$\hat{H}\hat{a} = \hat{a}\hat{H} - \hat{a}\hbar\omega$$

$$\hat{H}\hat{a}|n\rangle = (\hat{a}\hat{H} - \hat{a}\hbar\omega)|n\rangle$$
$$= \hat{a}\hat{H}|n\rangle - \hat{a}\hbar\omega|n\rangle$$
$$= E_n\hat{a}|n\rangle - \hbar\omega\hat{a}|n\rangle$$
$$= (E_n - \hbar\omega)\hat{a}|n\rangle$$

$\hat{a}|n\rangle$ be the eigen fun^n of hamiltonian with eigen value $(E_n - \hbar\omega)$.

Compare by $E_n = (n+\frac{1}{2})\hbar\omega$, we observe that above result is one state lower.

So a is Lowering Operater

$$\boxed{\hat{a}|n\rangle = (\quad)|n-1\rangle} \longrightarrow$$ This is not eigen value eq^n bcoz state changes, (decrease)

Action of lowering O/p $\Rightarrow$ To change the state by unity.

Now for a^+, $\hat{H}\hat{a}^+|n\rangle = (\hat{a}^+\hat{H} + \hat{a}^+\hbar\omega)|n\rangle$
$$= \hat{a}^+\hat{H}|n\rangle + \hat{a}^+\hbar\omega|n\rangle$$
$$= E_n\hat{a}^+|n\rangle + \hbar\omega\hat{a}^+|n\rangle$$
$$= (E_n + \hbar\omega)\hat{a}^+|n\rangle$$

Compare with $E_n = (n+\frac{1}{2})\hbar\omega$, this state is one state raised (greater)

So $\hat{a}^+$ is Raising operator

$$\boxed{\hat{a}^+|n\rangle = (\quad)|n+1\rangle}$$

Action of Raising O/p $\Rightarrow$ To rais (increase) the state by unity.

So We have, $\hat{a}|n\rangle = C_-|n-1\rangle$ ——— (1)

$\hat{a}^+|n\rangle = C_+|n+1\rangle$ ——— (2)

Now, we can find out the cofficients C_- & C_+.

Take Hermitian conjugate of eq^n (1),

$$\langle n|\hat{a}^{+} = C_-^{*} \langle n-1|$$

Now $\langle n|\hat{a}^{+}\hat{a}|n\rangle = C_-^{*}C_- \underbrace{\langle n-1|n-1\rangle}_{=1}$

$$\langle n|\hat{a}^{+}\hat{a}|n\rangle = |C_-|^2$$

$$\langle n|n|n\rangle = |C_-|^2$$

$$\Rightarrow n\langle n|n\rangle = |C_-|^2$$

$$\Rightarrow n = |C_-|^2 \qquad \{\langle n|n\rangle = 1\}$$

$$\Rightarrow C_- = \pm\sqrt{n}$$

Only + values are allowed. bcoz $|n-1\rangle$ is the state vector & C_- rep^n the magnitude of state vector i.e. Norm. & Norm can't be −ve.

So $\boxed{C_- = +\sqrt{n}}$

Similarly, $C_+ = \sqrt{n+1}$

So

$\hat{a}\|n\rangle = \sqrt{n}\|n-1\rangle$
$\hat{a}^{+}\|n\rangle = \sqrt{n+1}\|n+1\rangle$
$\hat{a}^{+}\hat{a}\|n\rangle = n\|n\rangle$

$\hat{a}^{+}\hat{a}$ is the Number operator.

<u>Matrix Element</u> for $\hat{a}$, $\hat{a}^{+}$ & $\hat{a}^{+}\hat{a}$,

$$\langle m|\hat{a}|n\rangle = \sqrt{n}\langle m|n-1\rangle$$
$$= \sqrt{n}\,\delta_{m,n-1}$$

$$\langle m|\hat{a}^{+}|n\rangle = \sqrt{n+1}\,\delta_{m,n+1}$$

$$\langle m|\hat{a}^{+}\hat{a}|n\rangle = n\,\delta_{mn}$$

By the action of $\hat{a}$ & $\hat{a}^{+}$, we can calculate the eigen fuc^n of 1 Dim linear H.O.

Operate the o/p on ground, first, second --- excited state then

$$\hat{a}^{+}|0\rangle = |1\rangle$$

$$\hat{a}^{+}|1\rangle = \sqrt{2}\,|2\rangle$$

$$\Rightarrow \cancel{\hat{a}^{+}}|2\rangle = \frac{\hat{a}^{+}|1\rangle}{\sqrt{2}} = \frac{(\hat{a}^{+})^{2}|0\rangle}{\sqrt{2!}}$$

for n^{th} state,

$$\hat{a}^{+}|n-1\rangle = \sqrt{n}\,|n\rangle$$

$$\Rightarrow |n\rangle = \underline{a^{+}|n-1\rangle}$$

If we know the lower state, then we can find out ~~next~~ excited state i.e. if we know ground state then we can calculate Ist excited state.

e.g.

$$\hat{a}|1\rangle = |0\rangle$$

$$\hat{a}|2\rangle = \sqrt{2}\,|1\rangle$$

$$|1\rangle = \frac{\hat{a}}{\sqrt{2}}|2\rangle$$

$$|0\rangle = \frac{(\hat{a})^{2}}{\sqrt{2!}}|2\rangle$$

$$\boxed{|0\rangle = \frac{(\hat{a})^{n}}{\sqrt{n!}}|n\rangle}$$

If we know lower state then by raising o/p, we can find higher state

" " " higher " " " lowering " " " lower ".

If we o/p lowering o/p on ground state then we get 0.

$$\boxed{\hat{a}|0\rangle = 0}$$

By Using this eqⁿ, ground state can be calculated.
IIly, if we o/p raising o/p on maximum highest possible state then we'll get 0. $n = 0, 1, \ldots \infty$ (states)

We have $\hat{a} = \left\{ \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{i\left(-i\hbar\frac{d}{dx}\right)}{\sqrt{2\hbar m\omega}} \right\}$

Now, $\left\{ \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{i\left(-i\hbar\frac{d}{dx}\right)}{\sqrt{2\hbar\omega m}} \right\} |0\rangle = 0$

$$\Rightarrow \left(\sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{\hbar\frac{d}{dx}}{\sqrt{2\hbar m\omega}} \right) \psi_0(x) = 0.$$

$$\frac{\hbar\frac{d}{dx}\psi_0(x)}{\sqrt{2\hbar m\omega}} = -\sqrt{\frac{m\omega}{2\hbar}}\,\hat{x}\,\psi_0(x)$$

$$\frac{d\psi_0(x)}{dx} = -\frac{m\omega}{\hbar}\hat{x}\,\psi_0(x)$$

By Integrating this differential eqⁿ, we can calculate $\psi_0(x)$.

$$\int \frac{d\psi_0(x)}{dx} = -\int \frac{m\omega}{\hbar}\hat{x}\,dx + c$$

$$\ln\psi_0 = -\frac{m\omega}{\hbar}\frac{\hat{x}^2}{2} + \ln A$$

$$\ln\left(\frac{\psi_0}{A}\right) = \frac{-m\omega}{2\hbar}\hat{x}^2$$

$$\boxed{\psi_0 = A\exp\left(\frac{-m\omega}{2\hbar}\hat{x}^2\right)}$$

This is the wave fucⁿ of ground state for L.H.O.

α^2 for L.H.O. is defined as,

$$\alpha^2 = \frac{m\omega}{\hbar}$$